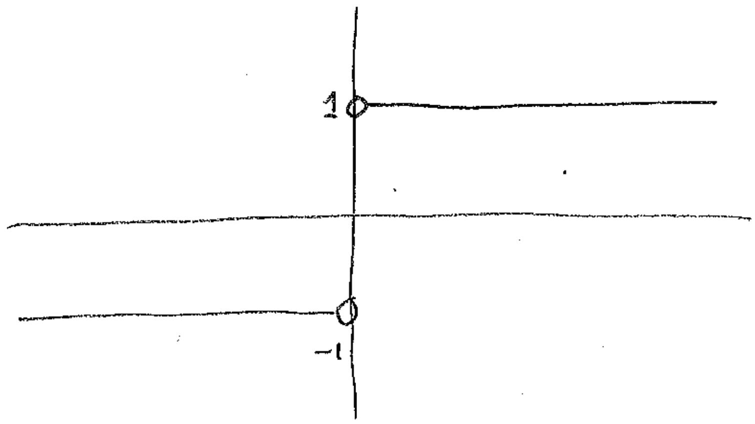
§ 2.4 - One - Sided limits

Motivation Example:

Consider the function $f(x) = \dfrac{x}{|x|}$ that has the graph

$$= \begin{cases} \dfrac{x}{x} = 1 , & x > 0 \\ \dfrac{x}{-x} = -1 , & x < 0 \end{cases}$$



What can you say about $\lim_{x \to 0} f(x)$? (Hint: Write a table!)

But notice,

1- If we approach 0 from the left (values less than zero) f(x) approaches -1, so we can say that the limit from the left is equal to -1 and we write it as

$$\lim_{x \to 0^-} f(x) = -1$$

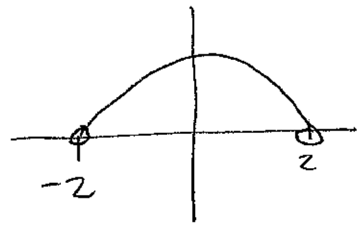
2- If we approach 0 from the right (values slightly greater than 0) f(x) approach 1, so we can say that the limit from the right is equal to 1 and we write it as $\lim_{x \to 0^+} f(x) = 1$.

Example 1:

(a) let $f(x) = \sqrt{4-x^2}$, Find

1- $\lim\limits_{x\to 2^+} \sqrt{4-x^2}$ DNE , because $x \to 2^+$ means $x > 2$!

2- $\lim\limits_{x\to 2^-} \sqrt{4-x^2} = 0$



3- $\lim\limits_{x\to -2^-} \sqrt{4-x^2} = 0$

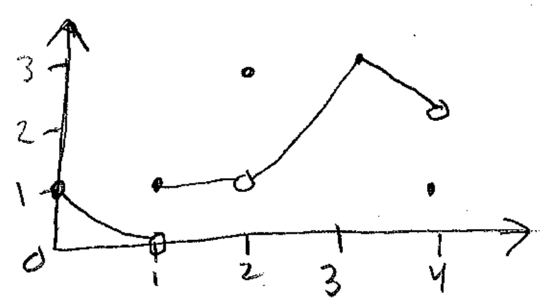
4- $\lim\limits_{x\to -2^+} \sqrt{4-x^2} = 0$

Theorem:

"If and only if"

$\lim\limits_{x\to c} f(x) \iff \lim\limits_{x\to c^+} f(x) = L$ and $\lim\limits_{x\to c^-} f(x) = L$.

Exercise: consider the function



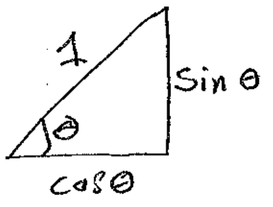
(a) $\lim\limits_{x\to 0} f(x)$ (b) $\lim\limits_{x\to 1} f(x)$ (c) $\lim\limits_{x\to 2} f(x)$ (d) $\lim\limits_{x\to 3} f(x)$

(d) $\lim\limits_{x\to 4} f(x)$

Exercise 2: Find the following limits

(a) $\lim\limits_{x\to 1^+} \sqrt{\dfrac{x-1}{x+2}}$ (b) $\lim\limits_{x\to -2^+} (x+3)\dfrac{|x+2|}{(x+2)}$.

Recall:



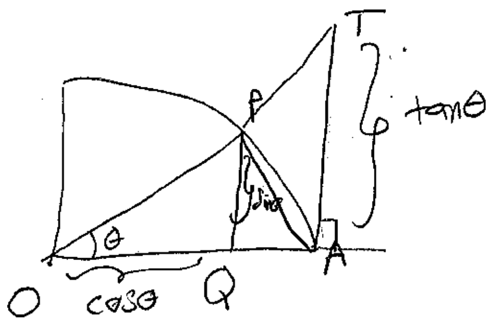
and $\tan\theta = \dfrac{\sin\theta}{\cos\theta} = \dfrac{\text{opposite}}{\text{adjacent}}$

Theorem:

$$\lim_{x\to 0} \frac{\sin\theta}{\theta} = 1$$

Proof:

we will just prove that $\lim\limits_{x\to 0^+} \dfrac{\sin\theta}{\theta} = 1$. consider the figure



$\tan\theta = \dfrac{\text{opposite}}{\text{adjacent}} = \dfrac{TA}{OA}$

$= TA$

Now $\underbrace{\text{area } \Delta OAP}_{①} \leq \underbrace{\text{area sector } OAP}_{②} \leq \underbrace{\text{area } \Delta OAT}_{③}$

① area $\Delta OAP = \frac{1}{2}(\text{base})(\text{height}) = \frac{1}{2}(OA)(PQ) = \frac{1}{2}(1)\sin\theta = \frac{\sin\theta}{2}$

② area sector $OAP = \frac{1}{2} r^2\theta = \frac{1}{2}(1)^2\theta = \frac{\theta}{2}$

③ area $\Delta OAT = \frac{1}{2}(\text{base})(\text{height}) = \frac{1}{2}(OA)(TA) = \frac{1}{2}\tan\theta$

hence,

$$\underbrace{\frac{1}{2}\sin\theta}_{①} \le \underbrace{\frac{1}{2}\theta}_{②} \le \frac{1}{2}\tan\theta \quad \text{— } (\ast)$$

$$\sin\theta \le \theta \le \frac{\sin\theta}{\cos\theta} \quad \text{— Divide by } \sin\theta$$

$$1 \le \frac{\theta}{\sin\theta} \le \frac{1}{\cos\theta}$$

$$1 \ge \frac{\sin\theta}{\theta} \ge \cos\theta$$

$$\lim_{x\to0^+} 1 \ge \lim_{x\to0^+}\frac{\sin\theta}{\theta} \ge \lim_{x\to0^+}\cos\theta$$

$$1 \ge \lim_{x\to0^+}\frac{\sin\theta}{\theta} \ge 1 \rightsquigarrow \boxed{\lim_{x\to0^+}\frac{\sin\theta}{\theta} = 1}$$

To show $\lim_{x\to0^-}\frac{\sin\theta}{\theta} = 1$, we need to know that the $f(\theta) = \frac{\sin\theta}{\theta}$ is even function! (finish the proof yourself!).

Example 2:

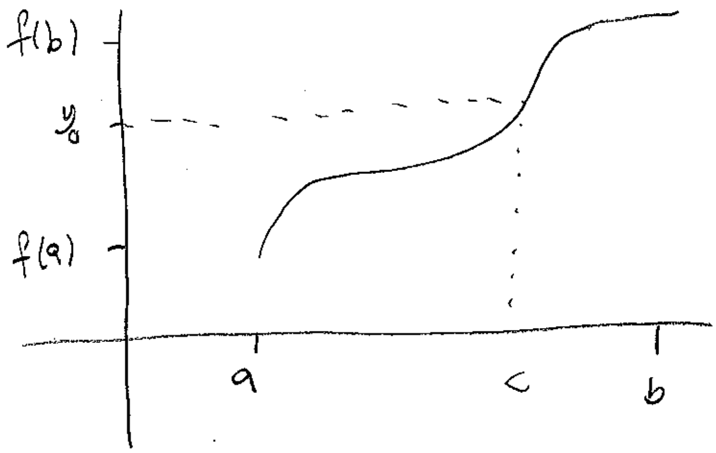
(a) $\lim_{x\to0}\frac{1-\cos x}{x}$, note that using double-angle formula $\frac{1-\cos x}{2} = \sin^2\frac{x}{2} \rightsquigarrow$

$$\lim_{x\to0}\frac{2\sin^2\frac{x}{2}}{x} = \lim_{x\to0} 2\underbrace{\frac{\sin\frac{x}{2}}{x}}_{=1}\cdot\sin\frac{x}{2} = 0.$$

(b) $\lim_{x\to0}\frac{\sin 2x}{5x}$ ≠ let $2x = y$ $\rightsquigarrow \lim_{x\to0}\frac{\sin y}{\frac{5y}{2}} = \frac{2}{5}\lim_{y\to0}\frac{\sin y}{y} = \frac{2}{5}$.

4- Intermediate Value Theorem for Continuous Functions

If f is continuous function on $[a,b]$ and if y_0 is any value between $f(a)$ and $f(b)$, then there exists c in $[a,b]$ such that $y_0 = f(c)$.

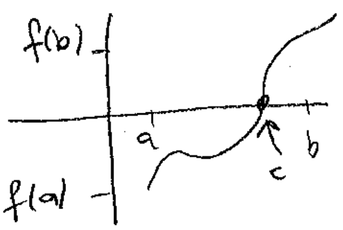


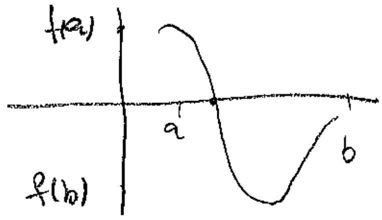
Application of the intermediate value Theorem

we will use the intermediate value theorem to find root (zeros) of functions. $f(x)=0$ on $[a,b]$

Strategy:

① check if $f(a)$ & $f(b)$ have different sign, so we can apply the IVT to find a point $c \in [a,b]$ such that $f(c)=0$.





② If $f(a)$ & $f(b)$ are from the same sign, look at two points in $[a,b]$ that have different sign and apply IVT to find a root between them.

Example: Show there exists a root for $x^3 - x - 1 = 0$ between 1 and 2.

Solution:

① $f(1) = 1^3 - 1 - 1 = -1$, $f(2) = 2^3 - 2 - 1 = 8 - 3 = 5$

Since $f(1)$ & $f(2)$ have different sign, we apply IVT to get a root in the middle of 1 & 2.

Exercise: Show there exists a root of each of the following:-

(a) $f(x) = x^3 - 3x - 1$

(b) $f(x) = \sqrt{2x+5} - 4 + x^2$.