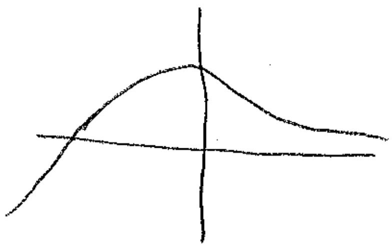
§2.5 - Continuity

Motivation question:

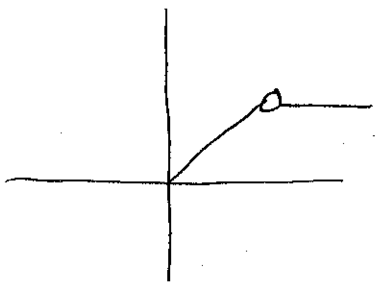
What is a continuous function?

Intutively, A function $y = f(x)$ is <u>continuous</u> function if we can sketch the graph of the function without <u>lifting</u> off the pencil. (Think of the graph as a railroad and a train will pass through it).

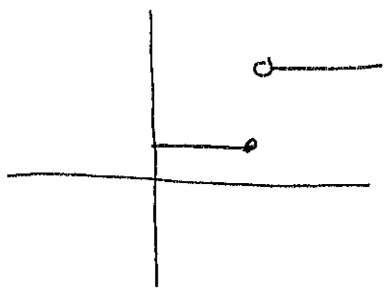
Example 1:



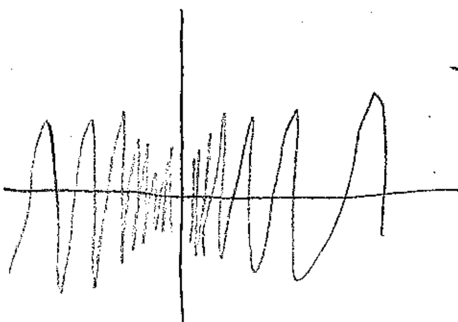
Continuous



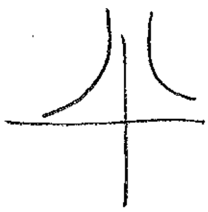
Not continuous
(A hole is removed)



Not continuous
(Jump)



Not continuous
(oscalitaing)



(infinite discontinuity)

To <u>Check</u> if a function is continuous

Geometry

graph the function and check if you can trace the graph <u>without</u> lifting the pencil

(disadvantage: Tedious to graph functions)

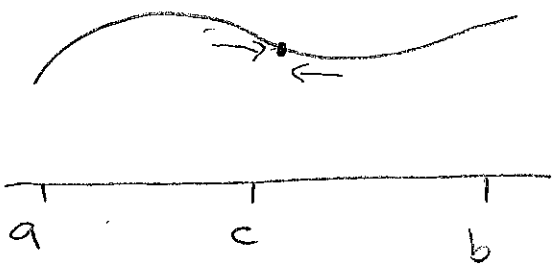
Algebra

$\boxed{\text{limits}}$

(easier)

1* Determining whether a function is continuous algebraically using the limit.

① Interior point



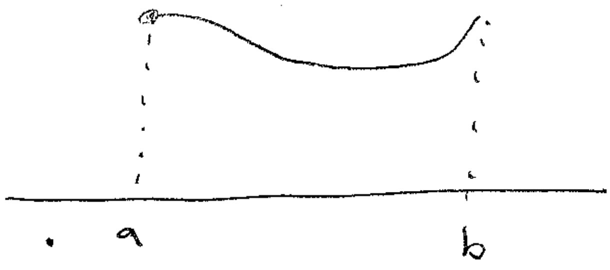
f is <u>continuous</u> at $x = c$ if

① $\lim_{x \to c} f(x)$ exist $\left(\lim_{x \to c^+} f(x) = \lim_{x \to c^-} f(x) \text{ exist} \right)$ ↙ No jump

② $f(c)$ exist (No oscillating).

③ $\lim_{x \to c} f(x) = f(c)$ (No holes)

② <u>Endpoint</u>



<u>Exercise</u> write the conditions for the continuity at a & b.

<u>Notation</u>:

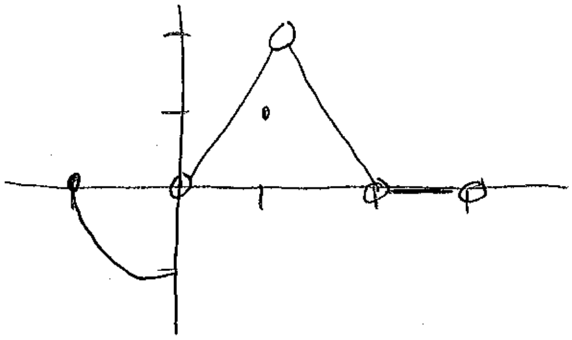
① If f is <u>not</u> continuous at $x = c$, we say f is discontinuous.

② f is left-continuous at $x = c$ if $\lim_{x \to c^-} f(x) = f(c)$

③ f is right-continuous at $x = c$ if $\lim_{x \to c^+} f(x) = f(c)$.

Exercise 1: consider the function

$$f(x) = \begin{cases} x^2-1 & , -1 \leq x < 0 \\ 2x & , 0 < x < 1 \\ 1 & , x=1 \\ -2x+4 & , 1 < x < 2 \\ 0 & , 2 < x < 3 \end{cases}$$



(a) Does $f(-1)$ exist?

(b) Is f continuous at $x=-1$?

(c) At $x=1$?

(d) At $x=2$?

(e) where it is continuous?

(f) What should be the value of $f(2)$ to be continuous?

(g) How about $x=1$?

Example 2: For what value of a is

$$f(x) = \begin{cases} x^2-1 & , x < 3 \\ 2ax & , x \geq 3 \end{cases}$$

continuous at every x?

Solution:

- If $x<3$, we have $f(x) = x^2-1$ (Polynomial and hence always continuous)
- If $x>3$, we have $f(x) = 2ax$ (— — — —)
- If $x=3$, we apply the continuity test immediately as there are changing.

To get f be continuous at $x=3$, we must have

$$\lim_{x\to 3^+} f(x) = \lim_{x\to 3^-} f(x) = f(3)$$

$$\lim_{x\to 3^+} x^2-1 = \lim_{x\to 3^-} 2ax$$

$$8 = 2a(3) \longrightarrow \boxed{a = \frac{8}{6} = \frac{4}{3}}$$

<u>Exercise</u>: Find the values of a such that f is continuous at every point.

$$f(x) = \begin{cases} a^2x - 2a & , x \geq 2 \\ 12 & , x < 2 \end{cases}$$

2- Continuous function

A function is <u>continuous on an interval</u> if it is continuous at <u>every</u> point of the interval.

<u>Theorem</u> If f, g are continuous, then so

$$f \pm g,\ f \cdot g,\ f^n,\ \sqrt[n]{f},\ \frac{f}{g}$$

<u>Example</u>:

1. Polynomial functions
2. $\sin x$, $\cos x$
3. $\dfrac{P(x) \leftarrow \text{polynomial}}{Q(x) \leftarrow \text{polynomial}}$ with $Q(c) \neq 0$ at <u>every</u> c.

<u>Theorem</u>: If f is continuous at c and g is continuous at $f(c)$, then the composite $g \circ f$ is continuous at c.

Example 3: Find the points where the following functions are continuous.

(a) $y = \frac{1}{x-2} - 3x$ (b) $y = \frac{1}{|x|+1} - \frac{x^2}{2}$

(c) $y = \sqrt{2x+3}$

Solution:

(a) In general, the function is continuous everywhere except where problems could occur (zero in the denominator, negative in the radical, values outside the domain of some functions).

So $y = \frac{1}{x-2} - 3x$ has a zero denominator at $x = 2$.

So the function is not continuous at $x = 2$.

(b) $y = \frac{1}{|x|+1} - \frac{x^2}{2}$ is not continuous where the denominator is zero, so we solve $|x| + 1 = 0 \rightarrow |x| = -1$ which has no solution, so there are no values in which the denominator is zero, so the function is continuous everywhere. ▩

(c) $y = \sqrt{2x+3}$, the function is continuous when $2x+3 \geq 0$

so when $x \geq -\frac{2}{3}$, i.e., on the interval $(-\frac{2}{3}, \infty)$.

3- limit of continuous Functions and extension of a point

Theorem:

assume g is continuous at b and $\lim_{x \to c} f(x) = b$, then

$$\lim_{x \to c} g(f(x)) = g\left(\lim_{x \to c} f(x)\right) = g(b).$$

Example:

(a) $\lim_{x \to 0} \sqrt{x^2+x+1} = \sqrt{\lim_{x \to 0} x^2+x+1} = \sqrt{0^2+0+1} = \sqrt{1} = 1.$

(b) $\lim_{x \to 1} \sqrt[3]{\frac{x-1}{x^2+1}} = \sqrt[3]{\lim_{x \to 1} \frac{x-1}{x^2+1}} = \sqrt[3]{0} = 0.$

Extension of a function

Example: consider the function $y = \frac{\sin x}{x}$. Is it continuous everywhere? How we can make it continuous everywhere?

It is not continuous at $x=0$ (zero denominator), so it is continuous everywhere except at $x=0$. To make it continuous, we must have $f(0) = \lim_{x \to 0} f(x) = \lim_{x \to 0} \frac{\sin x}{x} = 1$, so

extension of f

$$F(x) = \begin{cases} \frac{\sin x}{x} & , x \neq 0 \\ 1 & , x = 1 \end{cases}$$

Exercise: $f(x) = \frac{x^2+x-6}{x^2-4}$, $x \neq 2$, Find the extension of this function at $x=2$.