§ 2.6 - Limits involving Infinity ; Asymptotes of Graphs

1- Finite limits as $X \to \pm\infty$

2- Horizontal Asymptotes

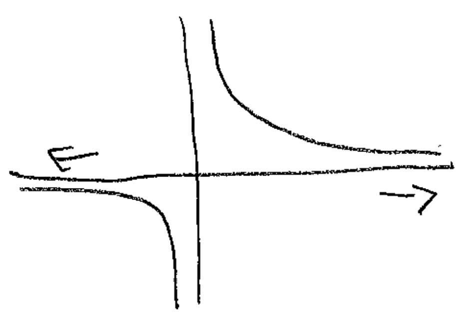
3- Infinite limits

4- Vertical Asymptotes

1- Finite limits as $X \to \pm\infty$

Example 1 : Consider $f(x) = \frac{1}{x}$. What happen if X is sufficiently large (i.e., X approaches ∞)? In other words, What is $\lim_{X\to\infty} \frac{1}{X}$ and $\lim_{X\to-\infty} \frac{1}{X}$?

Either, we see the graph or we make a table!



we can easily see that

$$\lim_{X\to\infty} \frac{1}{X} = 0 \quad \text{and} \quad \lim_{X\to-\infty} \frac{1}{x} = 0.$$

$\boxed{\text{General Rule}}$ $\lim_{X\to\pm\infty} \frac{1}{X^n} = 0$ if $n \geq 1$.

$$\lim_{X\to\pm\infty} \frac{1}{X^n} = \lim_{X\to\pm\infty} \frac{1}{X} \cdot \frac{1}{X} \cdot \frac{1}{X} \cdots \frac{1}{X} = \lim_{X\to\pm\infty} \frac{1}{X} \cdot \lim_{X\to\pm\infty} \frac{1}{X} \cdots \lim_{X\to\pm\infty} \frac{1}{X} = 0$$

limits at infinity of Rational Functions

① To find the the limit of rational function as $x \to \pm\infty$ we divide both the numerator and denominator by the highest power of x in the denominator.

Examples:

$\frac{\infty}{\infty}$ undefined!

(a) $$\lim_{x\to\infty} \frac{3x^2-x-2}{5x^2+4x+1} = \lim_{x\to\infty} \frac{\frac{3x^2}{x^2}-\frac{x}{x^2}-\frac{2}{x^2}}{\frac{5x^2}{x^2}+\frac{4x}{x^2}+\frac{1}{x^2}} = \lim_{x\to\infty} \frac{3-\frac{1}{x}-\frac{2}{x^2}}{5+\frac{4}{x}+\frac{1}{x^2}}$$

$$= \frac{\lim_{x\to\infty} 3 - \lim_{x\to\infty}\frac{1}{x} - \lim_{x\to\infty}\frac{2}{x^2}}{\lim_{x\to\infty} 5 + \lim_{x\to\infty}\frac{4}{x} + \lim_{x\to\infty}\frac{1}{x^2}} = \boxed{\frac{3}{5}}$$

(b) $$\lim_{x\to\infty} \frac{5x^2+8x-3}{3x^2}$$ (exercise)

(c) $$\lim_{x\to\infty} \frac{11x+2}{2x^3-1} = \lim_{x\to\infty} \frac{\frac{11x}{x^3}+\frac{2}{x^3}}{\frac{2x^3}{\cancel{x^3}}-\frac{1}{x^3}} = \lim_{x\to\infty} \frac{\frac{11}{x^2}+\frac{2}{x^3}}{2-\frac{1}{x^3}} = \frac{0+0}{2-0} = \frac{0}{2} = 0$$

② To find the limit of a rational function as $x \to \pm\infty$ take the highest power of x in the numerator and denominator as a common factor and proceed.

Example:

(a) $$\lim_{x\to\infty} \frac{3x+7}{x^2-2} = \lim_{x\to\infty} \frac{x(3+\frac{7}{x})}{x^2(1-\frac{2}{x^2})} = \lim_{x\to\infty} \frac{\cancel{x}(3+\frac{7}{x})}{x^{\cancel{2}}(1-\frac{2}{x^2})} = \frac{0(3+\cancel{0})}{(1-0)} = 0.$$

(b) $$\lim_{x\to\infty} \frac{\sqrt{2x^2+1}}{3x-5} = \lim_{x\to\infty} \frac{\sqrt{x^2\left(2+\frac{1}{x^2}\right)}}{x\left(3-\frac{5}{x}\right)} = \lim_{x\to\infty} \frac{\sqrt{x^2}\sqrt{2+\frac{1}{x^2}}}{x\left(3-\frac{5}{x}\right)}$$

$$= \lim_{x\to\infty} \frac{|x|\sqrt{2+\frac{1}{x^2}}}{x\left(3-\frac{5}{x}\right)} = \lim_{x\to\infty} \frac{\cancel{x}\sqrt{2+\frac{1}{x^2}}}{\cancel{x}\left(3-\frac{5}{x}\right)} = \frac{\sqrt{2}}{3}.$$

(c) $$\lim_{x\to-\infty} \frac{\sqrt{2x^2+1}}{3x-5}$$ (exercise)

(d) $$\lim_{x\to\infty}\left(\sqrt{x^2+1}-x\right) = \infty-\infty$$ (undefined!)

$$\lim_{x\to\infty} \frac{\left(\sqrt{x^2+1}-x\right)\left(\sqrt{x^2+1}+x\right)}{\left(\sqrt{x^2+1}+x\right)} = \lim_{x\to\infty} \frac{\cancel{x^2}+1-\cancel{x^2}}{\left(\sqrt{x^2+1}-x\right)} = \lim_{x\to\infty} \frac{1}{\sqrt{x^2+1}+x} =$$

$$= \lim_{x\to\infty} \frac{1}{x\left(\sqrt{1+\frac{1}{x^2}}+1\right)} = 0.$$

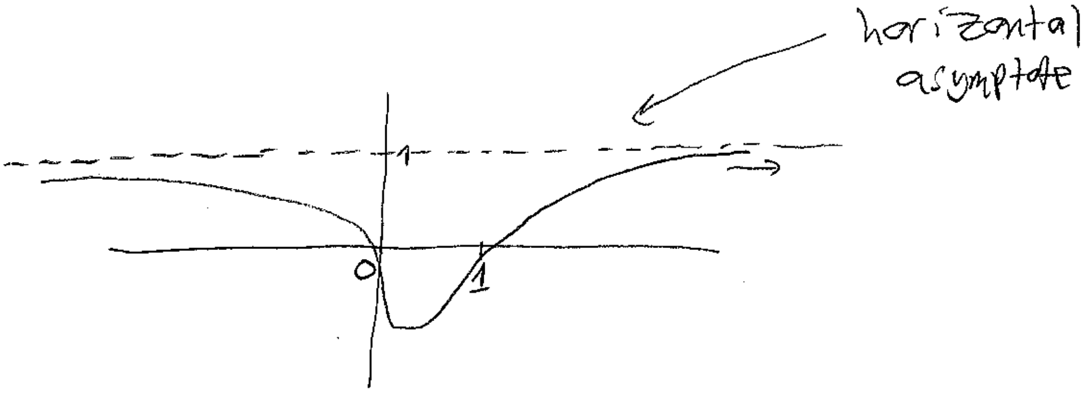
(d) $$\lim_{x\to\infty}\left(x-\sqrt{x^2+16}\right)$$ (exercise).

2- Horizontal Asymptote

The line $y=L$ is called horizontal asymptote of the curve $y=f(x)$ if either

$$\lim_{x\to\infty} f(x) = L \quad \text{or} \quad \lim_{x\to-\infty} f(x) = L.$$

Example: Consider the function $f(x) = \dfrac{x^2-1}{x^2+1}$, then we have seen that $\lim\limits_{x\to\infty} f(x) = 1$ and $\lim\limits_{x\to-\infty} f(x) = 1$, so the line $y = 1$ is a horizontal asymptote.



Example: Find the horizontal asymptote of the function

$$y = \frac{x-9}{\sqrt{4x^2+3x+2}}.$$

We need to find both $\lim\limits_{x\to\infty} f(x)$ and $\lim\limits_{x\to-\infty} f(x)$.

- $\lim\limits_{x\to\infty} f(x) = \lim\limits_{x\to\infty} \dfrac{x-9}{\sqrt{4x^2+3x+2}} = \lim\limits_{x\to\infty} \dfrac{x\left(1-\frac{9}{x}\right)}{\sqrt{x^2}\sqrt{4+\frac{3}{x}+\frac{2}{x^2}}} = \lim\limits_{x\to\infty} \dfrac{x\left(1-\frac{9}{x}\right)}{|x|\sqrt{4+\frac{3}{x}+\frac{2}{x^2}}}$

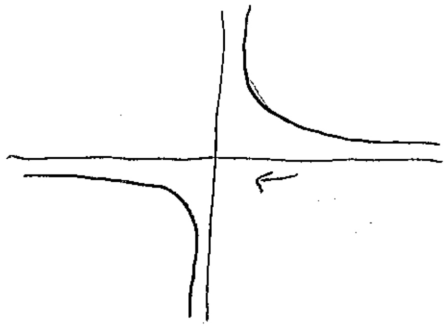
$$= \frac{1}{\sqrt{4}} = \frac{1}{2} \rightsquigarrow y = \frac{1}{2} \text{ is a horizontal asymptote}$$

- $\lim\limits_{x\to-\infty} f(x) = \lim\limits_{x\to-\infty} \dfrac{x-9}{\sqrt{4x^2+3x+2}} = -\dfrac{1}{2} \rightsquigarrow y = -\dfrac{1}{2}$ is a horizontal asymptote

3- Infinite limits

This time, we study limits that either be $+\infty$ or $-\infty$.

Example: what is $\lim\limits_{x\to 0^+} \frac{1}{x}$ and $\lim\limits_{x\to 0^-} \frac{1}{x}$?



$$\boxed{\lim_{x\to 0^+} \frac{1}{x} = \infty \quad \text{and} \quad \lim_{x\to 0^-} \frac{1}{x} = -\infty}$$

Side note:

$x \to 0^+$, then $\frac{1}{x} \to \infty$ } $x \to 0^-$, then $\frac{1}{x} \to -\infty$

Example: $\lim\limits_{x\to 1^+} \frac{1}{x-1}$ and $\lim\limits_{x\to 1^-} \frac{1}{x-1}$?

$x \to 1^+ \rightsquigarrow x-1 \to 0^+ \rightsquigarrow \frac{1}{x-1} \to \infty$.

$x \to 1^- \rightsquigarrow x-1 \to 0^- \rightsquigarrow \frac{1}{x-1} \to -\infty$.

Example:

(a) $\lim\limits_{x\to 2^+} \frac{(x+2)^2}{(x^2-4)} = \lim\limits_{x\to 2^+} \frac{(x+2)\cancel{(x+2)}}{(x-2)\cancel{(x+2)}} = \lim\limits_{x\to 2^+} \frac{x+2}{x-2} = \frac{4}{0^+} = \infty$.

This is for you, Don't write it in the exam

(b) $\lim\limits_{x\to 3} \frac{x}{x^2-9}$

$\lim\limits_{x\to 3^+} \frac{x}{x^2-9} = \frac{3}{0^+} = \infty$

$\lim\limits_{x\to 3^-} \frac{x}{x^2-9} = \frac{3}{0^-} = -\infty$

} $\lim\limits_{x\to 3} \frac{x}{x^2-9}$ Doesn't exist.

4- Vertical Asymptotes

A line $x = a$ is a vertical asympote of the graph of a function $y = f(x)$ if either

$$\lim_{x\to a^+} f(x) = \pm\infty \quad \text{or} \quad \lim_{x\to a^-} f(x) = \pm\infty$$

Example : Find the horizontal and vertical asymptotes of

(a) $y = \dfrac{x+3}{x+2}$

we are interested in the behaviour of $x \to \pm\infty$ and $x \to -2$ (since the denominator is zero).

$$\lim_{x\to\infty} \frac{x+3}{x+2} = \lim_{x\to\infty} \frac{x(1+\frac{3}{x})}{x(1+\frac{2}{x})} = \frac{1+0}{1+0} = 1 \rightsquigarrow \boxed{y = 1}$$ is a horizontal asymptote.

$$\lim_{x\to-\infty} \frac{x+3}{x+2} = \lim_{x\to-\infty} \frac{x(1+\frac{3}{x})}{x(1+\frac{2}{x})} = \frac{1+0}{1+0} = 1$$

$$\lim_{x\to 2^+} \frac{x+3}{x+2} = \frac{1}{0^+} = +\infty \rightarrow \boxed{x = 2}$$ is a vertical asymptote.

$$\lim_{x\to -2^-} \frac{x+3}{x+2} = \frac{1}{0^-} = -\infty$$

(b) (exercise) $f(x) = \dfrac{-8}{x^2-4}$