§3.10 - Related Rates

1- Derivative as rate of change.

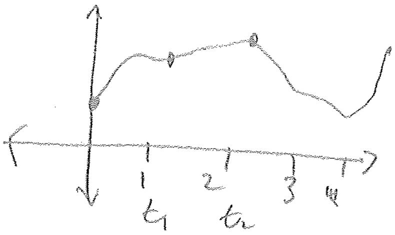
2- Related rates.

1- Derivative as rate of change

Consider a particle that moves according to the equation

$S = f(t)$, t is the time and S is the distance of the origin

Recall: speed $= \dfrac{\text{displacement}}{\text{time}} = \dfrac{f(t_2) - f(t_1)}{t_2 - t_1}$



↑
slope!

so if $t_2 \to t_1$, we get the derivative $S' = f'(t)$. so the derivative can be interpreted as velocity.

So we have

velocity $= v(t) = S'$ (m / second)

accelaration $= a(t) = S''$ (m / second^2)

Example 1: A particle is moving according to

$$S = t^3 - 12t^2 + 36t.$$

(1) Find the velocity at time t:

$$v(t) = s'(t) = 3t^2 - 24t + 36$$

(2) What is the velocity after 3 seconds?

$$v(3) = 3(3)^2 - 24(3) + 36 = -9 \ m/s^2 \ (\text{downward})$$

(3) When the particle is at rest.

$$v(t) = 0 \rightarrow 3t^2 - 24t + 36 = 0 \rightarrow t = 2 \text{ and } t = 8.$$

(4) What is the accelaration?

$$a(t) = v'(t) = 6t - 24.$$

2- Related Rates

Example 1: If $y = x^2$ and $\frac{dx}{dt} = 3$, what is $\frac{dy}{dt}$ when $x = -1$?

Solution:

Given: $\frac{dx}{dt} = 3$ and $x = -1$

Required: $\frac{dy}{dt}$ at $x = -1$

Relation: $y = x^2$

Differentiate both sides with respect to t, we get

$$\frac{dy}{dt} = 2x \cdot \frac{dx}{dt}$$

$$\left.\frac{dy}{dt}\right|_{x=-1} = 2(-1)(3) = -6.$$

Example 2: Air is being pumped in a spherical balloon so that its volume increases at a rate of 500 cm^3/second. How fast is the radius of the balloon increasing when the diameter is 50 cm?

(The volume $V = \frac{4}{3}\pi r^3$

Solution:

Given: $\frac{dV}{dt} = 500$ cm^3/second , $D = 50$ cm , so $r = \frac{50}{2} = 25$ cm

Required: $\frac{dr}{dt}$

Relation: $V = \frac{4}{3}\pi r^3$

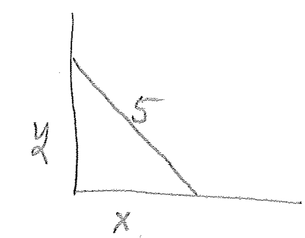
$$\frac{dV}{dt} = \frac{4}{3}\pi \cdot 3r^2 \cdot \frac{dr}{dt}$$

$$500 = \frac{4}{\cancel{3}}\pi \cdot \cancel{3}(25)^2 \frac{dr}{dt} \rightarrow 500 = 4(25)^2 \frac{dr}{dt} \rightarrow \frac{dr}{dt} = \frac{1}{5\pi} \text{ cm/sec}$$

Example 3: A ladder is 5 m long rests against a vertical wall. If the bottom of the ladder moves away at rate 0.5 m/ second. How fast is the top of the ladder glides down when the bottom of the ladder is 4 m from the wall?

Solution:

Given: $\frac{dx}{dt} = 0.5$ m/second , $x = 4$ m



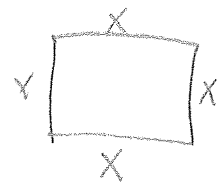
Required: $\frac{dy}{dt} = ?$ at $x = 4, y = 3$

Relation: $x^2 + y^2 = 25$

$$2x \cdot \frac{dx}{dt} + 2y\frac{dy}{dt} = 0 \rightsquigarrow \frac{dy}{dt} = -\frac{x}{y} \cdot \frac{dx}{dt}$$

$$= -\frac{4}{3} \cdot 4 = -\frac{16}{3} \text{ second/m.}$$

Example 4: Each side of a square is increasing at a rate 6 cm/second. At what rate of the area of the square increasin when the area of the square is 16 cm^2!



Given:

$$\frac{dx}{dt} = 6 \text{ cm/second}$$

Required:

$$\frac{dA}{dt} \text{ at } A = 16\text{ cm}^2 \quad (16 = x^2 \to x = 4)$$

Relation: $A = x^2$

$$\frac{dA}{dt} = 2x\frac{dx}{dt}$$

$$= 2(4)(3) = 24 \text{ cm}^2\text{/second.}$$

Exercise 1: The length ℓ of a rectangle is decreasing at the rate of 2 cm/second while the width w is increasing at the rate of 2 cm/sec. When $\ell = 12$ cm and $w = 5$ cm. Find the rate of change of

(a) area

(b) perimeter

(c) length of the diagonals.