§3.2 - The Derivative of a function

1- Slope, lines, and tangent lines

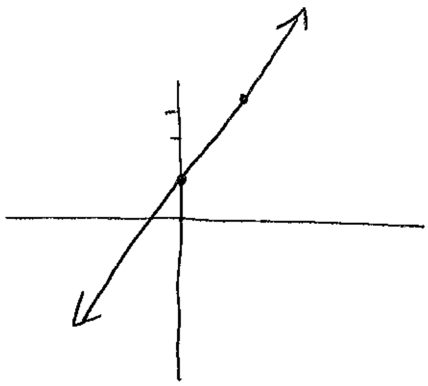
A **line** is a function of the form $y = \overset{\text{slope}}{m}x + b$.

- (To draw a line from its equation, find two points in the line and connect them).

Example: Consider the line $y = 2x+1$. Draw the graph of the line.

$x = 0 \longrightarrow y(0) = 2(0)+1 = 1 \longrightarrow (0,1)$

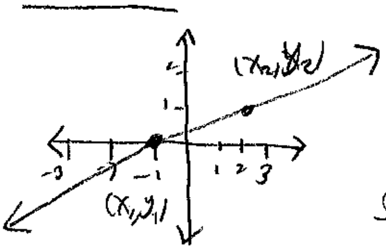
$x = 1 \longrightarrow y(1) = 2(1)+1 = 3 \longrightarrow (1,3)$



- To find the equation of the line from the graph, find the slope m and one point (x_1,y_1) on the line and substitute in the formula

$$y - y_1 = m(x - x_1) \quad , \quad m = \frac{y_2 - y_1}{x_2 - x_1}$$

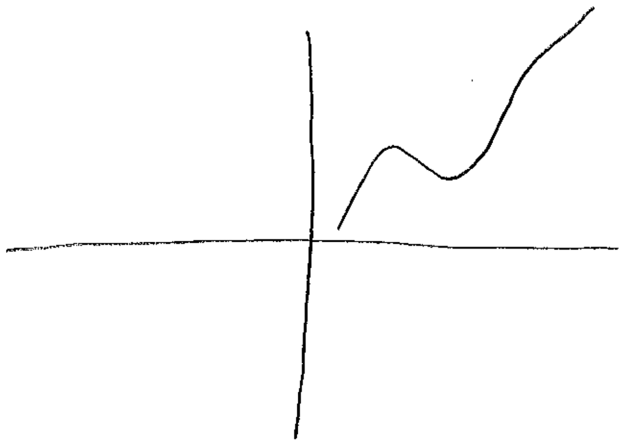
Example:



$$m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{1-0}{2-(-1)} = \frac{1}{3}$$

So $y - (0) = \frac{1}{3}(x - (-1)) \longrightarrow y = \frac{1}{3}(x+1) = \frac{1}{3}x + \frac{1}{3}$

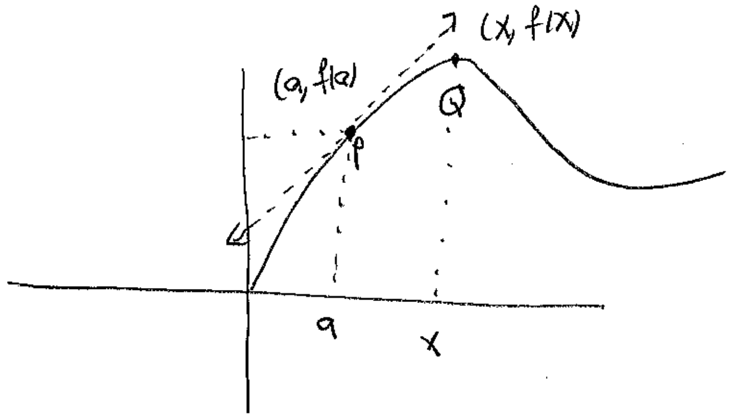
Tangent line :- Consider the graph of $y = f(x)$.



- Bisector line is a line that cut the graph into two points.
- Tangent line is a line that cut the graph in just one point.

2- Tangents and derivatives

Suppose we want to find the tangent line of the function $y = f(x)$ at the point $(a, f(a))$, so we need to find the slope



$$m_{PQ} = \frac{y_2 - y_1}{x_2 - x_1} = \frac{f(x) - f(a)}{x - a}$$

Now if Q approaches P we will get the slope of the tangent line.

$$\boxed{m = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}}$$

Example 3: Find ~~the~~ equation of the tangent line to $y = x^2$ at the point $(1, 1)$.

Solution: we have $a = 1$ and $f(x) = x^2$, the slope is

$$m = \lim_{x \to 1} \frac{f(x) - f(a)}{x - a} = \lim_{x \to 1} \frac{x^2 - 1^2}{x - 1} = \lim_{x \to 1} \frac{x^2 - 1}{x - 1} \left(= \frac{0}{0} \text{ undet} \right.$$

$$= \lim_{x \to 1} \frac{\cancel{(x-1)}(x+1)}{\cancel{(x-1)}} = \lim_{x \to 1} (x+1) = 2.$$

So the equation of the tangent line is

$$y - y_1 = m(x - x_1)$$

$$y - 1 = 2(x - 1)$$

$$y - 1 = 2x - 2$$

$$y = 2x - 2 + 1 \longrightarrow \boxed{y = 2x - 1}$$

Equivalent Definition:

let $x - a = h$, so we get if $x \to a$, hence $h \to 0$

$$m = \lim_{x \to a} \frac{f(x) - f(a)}{x - a} = \lim_{h \to 0} \frac{f(x+h) - f(a)}{h} \longleftarrow$$ usually easier to compute.

Example: Find ~~the~~ equation of the tangent line ~~to~~ the function $y = \frac{2}{x}$ at the point $x = 2$.

Solution: we have $a=2$ and $f(x)=\frac{2}{x}$ with $f(2)=\frac{2}{2}=1$

So the point is $(2,1)$. Next we need to find the slope of the tangent line

$$m = \lim_{h\to a}\frac{f(a+h)-f(a)}{h} = \lim_{h\to 0}\frac{f(2+h)-f(2)}{h} =$$

$$= \lim_{h\to 0}\frac{\frac{2}{2+h}-1}{h} = \lim_{h\to 0}\frac{\frac{2-(2+h)}{2+h}}{h}$$

$$= \lim_{h\to 0}\frac{-\cancel{h}}{\cancel{h}(2+h)} = \lim_{h\to 0}\frac{-1}{2+h} = \frac{-1}{2}.$$

So the tangent line has the equation

$$y-y_1=m(x-x_1)$$

$$y-1=-\frac{1}{2}(x-2) \longrightarrow y-1=\frac{-1}{2}x+1$$

$$\boxed{y=-\frac{1}{2}x+2}$$

3- Derivative of a function using the Definition

Recall:

Derivative is used to find the slope of a tangent line at a point. Hence

derivative

$$f'(a) = \text{slope of a tangent line}$$

$$= \lim_{x \to a} \frac{f(x) - f(a)}{x - a} = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}.$$

Example: Find the derivative of the function $f(x) = x^2 - 2x + 7$ at the point a.

Solution:

$$f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h} = \lim_{h \to 0} \frac{\left[(a+h)^2 - 2(a+h) + 7\right] - \left[a^2 - 2a + 7\right]}{h}$$

$$= \lim_{h \to 0} \frac{\left[\cancel{a^2} + 2ah + h^2 - \cancel{2a} - 2h + \cancel{7}\right] - \cancel{a^2} + \cancel{2a} - \cancel{7}}{h}$$

$$= \lim_{h \to 0} \frac{\cancel{h}\,(2a + h - 2)}{\cancel{h}} = \lim_{h \to 0} 2a + h - 2 = \boxed{2a - 2}$$

Definition

The derivative of a function is given by

$$\boxed{f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h}}$$

Example: Find using the definition, the derivative of the following functions

(a) $f(x) = x^3 + 2x$ (b) $f(x) = \sqrt{x+1}$ (c) $f(x) = \frac{x+1}{x+2}$

Solution:

(a) $$f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} = \lim_{h\to 0} \frac{(x+h)^3+2(x+h) - x^3-2x}{h}$$

$$= \lim_{h\to 0} \frac{\cancel{x^3}+3x^2h+3xh^2+h^3+\cancel{2x}+2h-\cancel{x^3}-\cancel{2x}}{h} = \lim_{h\to 0} \frac{\cancel{h}(3x^2+3xh+h^2+2)}{\cancel{h}}$$

$$= \lim_{h\to 0} 3x^2+3xh+h^2+2 = 3x^2+2.$$

(b) $$f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} = \lim_{h\to 0} \frac{\sqrt{x+h+1}-\sqrt{x+1}}{h} \cdot \frac{(\sqrt{x+h+1}+\sqrt{x+1})}{(\sqrt{x+h+1}+\sqrt{x+1})}$$

$$= \lim_{h\to 0} \frac{\cancel{x}+h+\cancel{1}-\cancel{x}-\cancel{1}}{h(\sqrt{x+h+1}+\sqrt{x+1})} = \lim_{h\to 0} \frac{\cancel{h}}{\cancel{h}(\sqrt{x+h+1}+\sqrt{x+1})} = \frac{1}{\sqrt{x+1}+\sqrt{x+1}}$$

$$= \frac{1}{2\sqrt{x+1}}.$$

Definition :

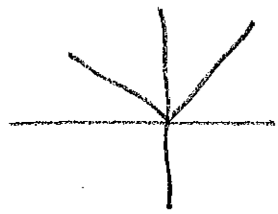
. A function is <u>differentiable at a</u> if $f'(a)$ exist
limit

. A function is <u>differentiable</u> on an interval (a,b) if it is differentiable at every point in the interval.

Question :

Can a function be not <u>differentiable</u>? Yes, because ~~Not~~ $f'(a)$ is a limit and the limit is not always exist.

Example : $f(x) = |x| = \begin{cases} x & , x \geq 0 \\ -x & , x < 0 \end{cases}$



. If $x>0$, then $f(x) = x$ differentiable.

. If $x<0$, then $f(x) = -x$ differentiable.

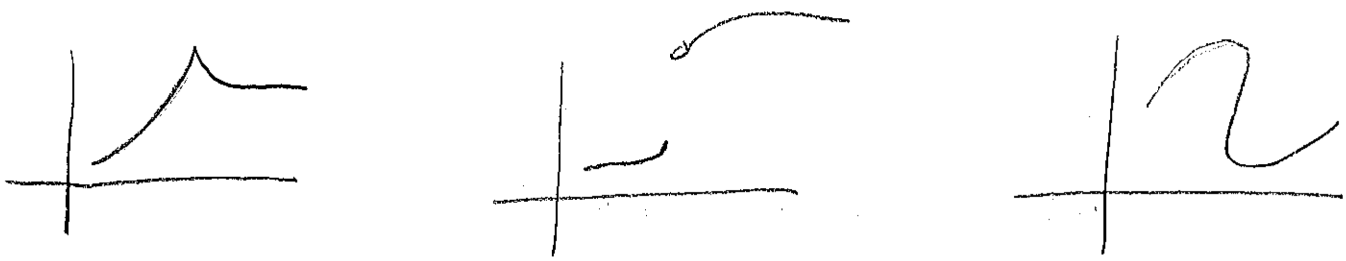
. If $x=0$, then we need to find $f'(0)$.

$$f'(0) = \lim_{x \to 0} \frac{f(x) - f(0)}{x - 0} = \lim_{x \to 0} \frac{f(x) - 0}{x}$$

$$\lim_{x \to 0^+} \frac{f(x)}{x} = 1 \qquad \lim_{x \to 0^-} \frac{f(x)}{x} = -1$$

So $\lim_{x \to 0} \frac{f(x)}{x}$ doesn't exist and so $f'(0)$ doesn't exist.

Situation where the function is not differentiable



(a) Corne (b) discontinuity (c) vertical tangent

Theorem: If f is differentiable, then it is continuous.

Proof:

we assume f is differentiable, i.e., $f'(a)$ exist at $x=a$

we have to show that $\lim_{x\to a} f(x) = f(a)$.

Note $$f(x) - f(a) = \frac{f(x) - f(a)}{(x-a)}(x-a)$$, take the limit

$$\lim_{x\to a}[f(x) - f(a)] = \lim_{x\to a}\frac{f(x) - f(a)}{(x-a)} \lim_{x\to a}(x-a)$$

$$\lim_{x\to a} f(x) - f(a) = f'(a) \cdot 0 \rightarrow \lim_{x\to a} f(x) = f(a)$$

4- Other Notation & higher Derivative $y = f(x)$

$$f'(x) = \frac{dy}{dx} = y' = \frac{df}{dx} = Df(x)$$, $f''(x)$ Second derivative

$f^{(n)}(x)$ nth, $f'''(x)$ Third derivative