§ 3.3 - Differentiation Rule

1. Differentiation Formula for special functions.
2. Differentiation rules for

1- Differentiation Formula for special functions

A- Constant Function $f(x) = c$, c is a constant.

$$\boxed{\frac{d}{dx}(c) = 0}$$

$$\frac{d}{dx}(f(x)) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} = \lim_{h\to 0} \frac{c-c}{h} = \lim_{h\to 0} \frac{0}{h} = \lim_{h\to 0} 0 = 0.$$

B- Power Function $f(x) = x^n$, n is a positive integer

$$\boxed{\frac{d}{dx}(x^n) = n\,x^{n-1}}$$

Recall: $z^2 - x^2 = (z-x)(z+x)$

$$z^3 - x^3 = (z-x)(\underbrace{z^2 + zx + x^2}_{})$$

3 - terms & each has total power of 2.

$$z^4 - x^4 = (z-x)(\underbrace{z^3 + z^2x + zx^2 + x^3}_{})$$

4-terms & each has total power of 3.

In general,

$$z^n - x^n = (z-x)(z^{n-1} + z^{n-2}x + z^{n-3}x^2 + \cdots + zx^{n-2} + x^{n-1})$$

So

$$f'(x) = \lim_{z \to x} \frac{f(z) - f(x)}{z - x} = \lim_{z \to x} \frac{z^n - x^n}{z - x} = \lim_{z \to x} \frac{\cancel{(z-x)}(z^{n-1} + z^{n-2}x + \cdots + zx^{n-2} + x^{n-1})}{\cancel{(z-x)}}$$

$$= \lim_{z \to x} z^{n-1} + z^{n-2}x + z^{n-3}x^2 + \cdots + zx^{n-2} + x^{n-1} = x^{n-1} + x^{n-2}x + x^{n-3}x^2 + \cdots + xx^{n-2} + x^{n-1}$$

$$= \underbrace{x^{n-1} + x^{n-1} + \cdots + x^{n-1}}_{n\text{-times}} = n x^{n-1}$$

- $\frac{d}{dx}(x^3) = 3x^2$, $\frac{d}{dx}(x^{15}) = 15x^{14}$

C- General power Rule $f(x) = x^n$, n is any real number.

$$\boxed{\frac{d}{dx}(x^n) = n x^{n-1}}$$

- $\frac{d}{dx}(\sqrt{x}) = \frac{d}{dx}(x^{\frac{1}{2}}) = \frac{1}{2} x^{\frac{1}{2}-1} = \frac{1}{2} x^{-\frac{1}{2}} = \frac{1}{2\sqrt{x}}$

- $\frac{d}{dx}(x^{\pi}) = \pi x^{\pi - 1}$

- $\frac{d}{dx}\left(\frac{1}{\sqrt[3]{x^2}}\right) = \frac{d}{dx}(x^{-\frac{2}{3}}) = -\frac{2}{3} x^{-\frac{2}{3}-1}$
$= -\frac{2}{3} x^{-\frac{5}{3}}$

D- Exponential Functions $f(x) = a^x$, a is ~~posi~~ positive number.

$$\boxed{\frac{d}{dx}(a^x) = a^x \cdot \ln a}$$

$$\frac{d}{dx}(a^x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} = \lim_{h\to 0} \frac{a^{x+h}-a^x}{h} = \lim_{h\to 0} \frac{a^x \cdot a^h - a^x}{h} = \lim_{h\to 0} a^x \cdot \frac{a^h-1}{h}$$

$$= a^x \cdot \underbrace{\lim_{h\to 0} \frac{a^h-1}{h}}_{=\ln a} = a^x \ln a.$$

(we will see this later).

Special case, let $a = e = 2.71878\cdots$ (Euler number), then

$$\frac{d}{dx}(e^x) = e^x \underbrace{\ln e}_{=1} = e^x.$$

2- New Derivative from old (Differentiation Rule).

1- Constant Multiple Rule. $\frac{d}{dx}(c f(x)) = c \cdot \frac{d}{dx}(f(x))$.

2- Sum/Difference Rule

$$\frac{d}{dx}(f(x) \pm g(x)) = \frac{d}{dx}(f(x)) \pm \frac{d}{dx}(g(x)).$$

Example 1: Find the derivative

(1) $\frac{d}{dx}(6x^2 - 10x - 5x^{-7}) = 6(2x) - 10 - 5(-7)x^{-7-1}$

$= 12x - 10 + 30x^{-8}$

(2) $\frac{d}{d\theta}\left(\frac{12}{\theta} - \frac{5}{\theta^4} + \frac{1}{\theta^7} + \theta^2\right) = \frac{d}{d\theta}(12\theta^{-1} - 5\theta^{-4} + \theta^{-7} + \theta^2) = 12(-1)\theta^{-2} - 5(-4)\theta^{-5}$
$+ (-7)\theta^{-8} + 2\theta$ $\boxed{4}$

$$3- \frac{d}{d\phi}\left(2\left(\frac{1}{\sqrt{r}} - r\right)\right) = 2\frac{d}{dr}\left(r^{-\frac{1}{2}} - r\right) = 2\left[-\frac{1}{2}r^{-\frac{1}{2}-1} - 1\right]$$

$$= -r^{+\frac{3}{2}} - 2.$$

Exercise 1:

1- $\frac{d}{dx}(3e^x)$.

2- $\frac{d}{dx}\left(\frac{x^4}{3} - \frac{2}{3}x^2 + 1\right)$.

3- $\frac{d}{dx}\left(x^{\frac{9}{5}} + \cancel{6}e^x\right)$.

3- Product Rule: (Leibniz rule)

$$\frac{d}{dx}\left(f(x)\cdot g(x)\right) = \frac{d}{dx}(f(x))\cdot g(x) + f(x)\cdot\frac{d}{dx}(g(x)).$$

4- Quotient Rule:

$$\frac{d}{dx}\left(\frac{f(x)}{g(x)}\right) = \frac{g(x)\frac{d}{dx}(f(x)) - f(x)\frac{d}{dx}(g(x))}{[g(x)]^2} = \frac{g(x)f'(x) - f(x)g'(x)}{[g(x)]^2}$$

Homework: proof using the derivative of the limits the rules 1 — 4.

Example 2: Find the following derivatives.

1- $\frac{d}{dx}\left(\underbrace{\sqrt{x}}_{f(x)}\,\underbrace{(3+2x)}_{g(x)}\right) = f'(x)\,g(x) + f(x)\,g'(x)$

$$= \frac{1}{2\sqrt{x}}(3+2x) + \sqrt{x}\,(2)$$

$$= \frac{3}{2\sqrt{x}} + \frac{x}{\sqrt{x}} + 2\sqrt{x} = \frac{3}{2\sqrt{x}} + \sqrt{x} + 2\sqrt{x}$$

$$= \frac{3}{2\sqrt{x}} + 3\sqrt{x}.$$

2- $\frac{d}{dx}\left(\underbrace{x}_{f(x)}\,\underbrace{e^{x}}_{g(x)}\right) = f'(x)\,g(x) + f(x)\,g'(x)$

$$= 1\cdot e^{x} + x\cdot e^{x} = e^{x} + xe^{x}.$$

3- $\frac{d}{dx}\left(\underbrace{(x-2)}_{f(x)}\,\underbrace{(x^2+5x-5)}_{g(x)}\right) = f'(x)\,g(x) + f(x)\,g'(x)$

$$= 1\cdot(x^2+5x-5) + (x-2)\cdot(2x+5)$$

$$= x^2+5x-5 + 2x^2 + x - 10 = 3x^2+6x-15.$$

4- $\frac{d}{dx}\left(\frac{\overbrace{x^3+5}^{f(x)}}{\underbrace{x}_{g(x)}}\right) = \frac{g(x)\,f'(x) - g'(x)\,f(x)}{[g(x)]^2} = \frac{x(3x^2) - (1)(x^3+5)}{[x]^2}$

$$= \frac{3x^3 - x^3 - 5}{x^2} = \frac{2x^3-5}{x^2}.$$

5- $\frac{d}{dx}\left(\frac{\sqrt{x}+x}{x^2}\right) = \frac{d}{dx}\left(\frac{\sqrt{x}}{x^2}+\frac{x}{x^2}\right) = \frac{d}{dx}\left(x^{-\frac{3}{2}}+x^{-1}\right) = -\frac{3}{2}x^{-\frac{5}{2}} - x^{-2}$.

6. $\frac{d}{dx}\left(\frac{\overbrace{x^3}^{f(x)}}{\underbrace{1-x^2}_{g(x)}}\right) = \frac{g'(x)f(x) - g(x)f'(x)}{[g(x)]^2} = \frac{(-2x)\cdot x^3 - (1-x^2)(3x^2)}{[1-x^2]^2}$

$= \frac{-2x^4 - 3x^2 + 3x^4}{[1-x^2]^2}$

Exercise 2: Differentiate the function.

1- $f(x) = \frac{x+5}{3x+7}$

2- $f(x) = 3x^{\frac{1}{2}} + e^4$

3- $f(x) = \frac{e^x}{x}$

4- $f(x) = \frac{ax+b}{cx+d}$

Example 3: Find an equation of the tangent line to the curve $y = \frac{\sqrt{x}}{2+x^2}$ at $x = 1$.

we need to find the slope $m_{tangent} = f'(1) = -\frac{1}{18}$

$f'(x) = \frac{g(x)f'(x) - f(x)g'(x)}{[g(x)]^2}$

$= \frac{(2+x^2)\left(\frac{1}{2\sqrt{x}}\right) - \sqrt{x}(2x)}{[2+x^2]^2}$

$= \frac{-1}{2(9)} = -\frac{1}{18}$

$y - y_1 = m(x - x_1)$

$y - \frac{1}{3} = -\frac{1}{18}(x-1)$

$-18y + 6 = x - 1 \sim \boxed{x + 18y = 7}$