§ 3.5 - Derivatives of Trigonometric Functions

1- Review of the trigonometric Functions.

2- Derivative of the basic trigonometric Functions

3- Derivatives of Functions that involve trigonometric functions.

1- Review of the trigonometric Functions

- Radian and degree.
- Definition of sine and cosine functions and their graphs.
- Definition of the other trigonometric functions.
- Some important identities.

- Angles are given by their degree, $30^\circ, 45^\circ, 60^\circ, \ldots$ as real numbers, we express them in terms of <u>radian</u>.

To convert from degree to radian:

$$\frac{\text{degree}}{180^\circ} \times \pi$$

To convert from radian to degree, replace π by 180°.

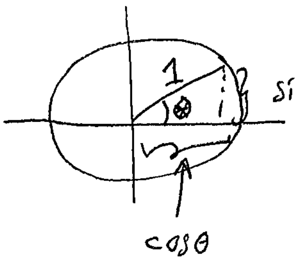
Example: Fill the table

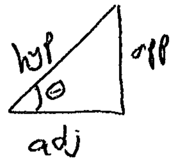
degree	0°	30°		60°	90°	120°		180°	270°	360°
radian			$\frac{\pi}{4}$		$\frac{\pi}{2}$		$\frac{3\pi}{4}$			

So 2π is a full circle and the angle repeats itself.

Definition of sine and cosine with their Graphs

Unit-circle | Right triangle





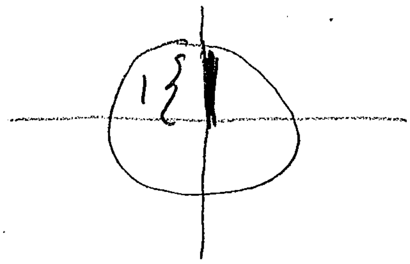
$\sin\theta = \dfrac{opp}{hypo}$

$\cos\theta = \dfrac{adj}{hypo}$

So $-1 \leq \sin\theta, \cos\theta \leq 1$

Example 0.2: $\sin\frac{\pi}{2}$, $\cos\frac{\pi}{2}$, $\sin\pi$, $\cos\pi$,

Recall that ~~Sin~~ $\frac{\pi}{2} = 90^\circ$

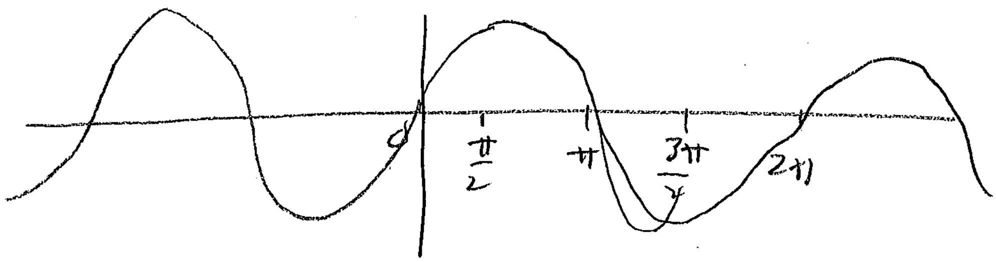


$\sin\frac{\pi}{2} = 1$ and $\cos\frac{\pi}{2} = 0$

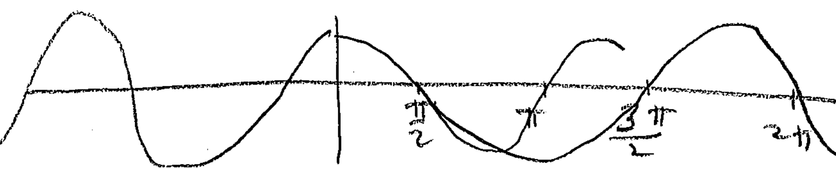
$\sin\pi = 0$, $\cos\pi = -1$

θ	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$	$\frac{3\pi}{4}$	$\frac{5\pi}{6}$	π	$\frac{3\pi}{2}$	2π
$\sin\theta$	0	$\frac{1}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{\sqrt{3}}{2}$	1	$\frac{\sqrt{3}}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{1}{2}$	1	-1	0
$\cos\theta$	1	$\frac{\sqrt{3}}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{1}{2}$	0	$-\frac{1}{2}$	$-\frac{1}{\sqrt{2}}$	$-\frac{\sqrt{3}}{2}$	-1	0	1

$y = \sin x$



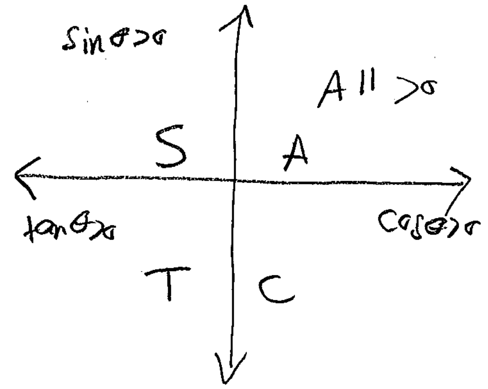
$y = \cos x$



• Definition of the other trigonometric Functions

$$\tan\theta = \frac{\sin\theta}{\cos\theta}, \quad \cot\theta = \frac{\cos\theta}{\sin\theta}$$

$$\sec\theta = \frac{1}{\cos\theta}, \quad \csc\theta = \frac{1}{\sin\theta}$$



"All students take Calculus"

• Some important Identities

1- $\sin^2\theta + \cos^2\theta = 1$

2- $\sec^2\theta = 1 + \tan^2\theta \quad (\csc^2\theta = 1 + \cot^2\theta)$

3- $\sin(-\theta) = -\sin\theta$ -- odd function and $\cos(-\theta) = \cos\theta$.

4. Double-angle formula:

$$\sin 2\theta = 2\sin\theta\cos\theta$$

$$\cos^2 2\theta = \cos^2\theta - \sin^2\theta$$

5. $\sin(a+b) = \sin a\cos b + \cos a\sin b$

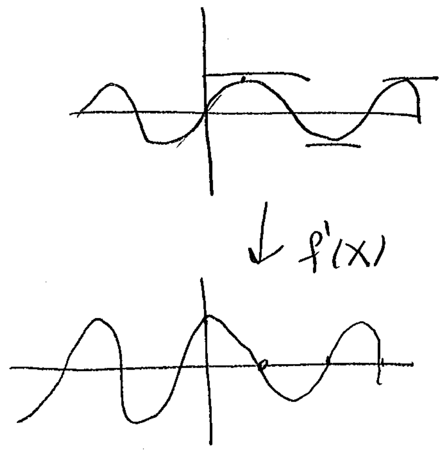
$\cos(a+b) = \cos a\cos b - \sin a\sin b$

will be useful for finding the derivative later!

2. Derivative of the trigonometric functions

Example 1:

$$\frac{d}{dx}(\sin x) = \cos x$$



Proof: Let $f(x) = \sin x$

$$\frac{d}{dx}(f(x)) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h}$$

$$= \lim_{h\to 0} \frac{\sin(x+h)-\sin x}{h} = \lim_{h\to 0} \frac{\sin x\cos h + \cos x \sin h - \sin x}{h}$$

$$= \lim_{h\to 0} \frac{\sin x(\cos h - 1) + \cos x \sin h}{h} = \lim_{h\to 0} \frac{\sin x(\cos h - 1)}{h} + \frac{\cos x \sin h}{h}$$

$$= \sin x \cdot \underbrace{\lim_{h\to 0} \frac{\cos h - 1}{h}}_{=0} + \cos x \cdot \underbrace{\lim_{h\to 0} \frac{\sin h}{h}}_{=1} = \cos x$$

Exercise 1: Prove that $\frac{d}{dx}(\cos x) = -\sin x$.

Example 2:

$$\frac{d}{dx}(\tan x) = \sec^2 x.$$

Proof:

$$\frac{d}{dx}(\tan x) = \frac{d}{dx}\left(\frac{\sin x}{\cos x}\right) = \frac{-(\cos x)'\sin x + \cos x(\sin x)'}{\cos^2 x} = \frac{+\sin^2 x + \cos^2 x}{\cos^2 x}$$

$$= \frac{\cos^2 x + \sin^2 x}{\cos^2 x} = \frac{1}{\cos^2 x} = \sec^2 x$$

$\blacksquare$

Exercise 2: Prove $\frac{d}{dx}(\cot x) = -\csc^2 x$.

Example 3:

$$\frac{d}{dx}(\sec x) = \sec x \tan x$$

Proof:

$$\frac{d}{dx}(\sec x) = \frac{d}{dx}\left(\frac{1}{\cos x}\right) = \frac{\cos x (1)' - 1(\cos x)'}{\cos^2 x} = \frac{0 - (-\sin x)}{\cos^2 x}$$

$$= \frac{\sin x}{\cos^2 x} = \underbrace{\frac{1}{\cos x}}_{\sec x} \cdot \underbrace{\frac{\sin x}{\cos x}}_{\tan x} = \sec x \tan x$$

$\blacksquare$

Exercise 3: Prove $\frac{d}{dx}(\csc x) = -\csc x \cot x$.

Summary

1. $\frac{d}{dx}(\sin x) = \cos x$
2. $\frac{d}{dx}(\cos x) = -\sin x$.
3. $\frac{d}{dx}(\tan x) = \sec^2 x$
4. $\frac{d}{dx}(\cot x) = -\csc^2 x$
5. $\frac{d}{dx}(\sec x) = \sec x \tan x$
6. $\frac{d}{dx}(\csc x) = -\csc x \cot x$.

3. Derivative of functions that involve trigonometric Functions

Example 4: Differentiate $y = x^2 - \cos x$

$$y' = 2x - (-\sin x) = 2x + \sin x.$$

Example 5: Find y' if

(1) $y = x^5 \sin x$

(2) $y = \dfrac{\sec x}{1+\sec x}$

(3) $y = \sqrt{x}\, \sec x.$

(4) $y = (\sin x + \cos x)\sec x$

(5) $y = x\cos x - \sin x.$

(6) $y = \tan x \cot x.$

(7) $y = \dfrac{4}{\cos x} + \dfrac{5}{\tan x}.$

Solution:

(1) we use the product rule,

$$y' = 5x^4 \sin x + x^5 \cos x$$

(2) we use the quotient rule,

$$y' = \frac{(1+\sec x)(\sec x)' - (\sec x)(1+\sec x)'}{(1+\sec x)^2} = \frac{(1+\sec x)\sec x \tan x - \sec x(\sec x \tan x)}{(1+\sec x)^2}$$

$$= \frac{\sec x \tan x + \sec^2 x \tan x - \sec^2 x \tan x}{(1+\sec x)^2} = \frac{\sec x \tan x}{(1+\sec x)^2}.$$

(3) $y = \sqrt{x}\,\sec x$

$y' = \dfrac{1}{2\sqrt{x}}\sec x + \sqrt{x}\,\sec x \tan x.$

(4) $y = (\sin x + \cos x)\sec x) = \sin x \sec x + \cos x \sec x$

$= \dfrac{\sin x}{\cos x} + \dfrac{\cos x}{\cos x} = \tan x + 1$

$y' = \sec^2 x$

(5) $y = x\cos x - \sin x$

$y' = -x\sin x + \cos x - \cos x = -x\sin x.$

(6) $y = \tan x \cot x = \dfrac{\sin x}{\cos x}\cdot\dfrac{\cos x}{\sin x} = 1$

$y' = 0.$

(7) $y = \dfrac{4}{\cos x} + \dfrac{5}{\sin x} = 4\sec x + 5\csc x$

$y' = 4\sec x \tan x - 5\csc x \cot x.$