§ 3.6 - The Chain Rule

Recall: composite of functions

$$(f \circ g)(x) = \underset{\text{outer function}}{f}\,[\underbrace{g(x)}_{\text{inner function}}]$$

Example 1: let $f(x) = x^3$, $g(x) = x^2 - 1$, then

$$(f \circ g)(x) = f(g(x)) = (x^2 - 1)^3$$

Goal: we want to differentiate $(f \circ g)(x) = f(g(x))$

Chain Rule:

$$\frac{d}{dx}(f(g(x))) = \underbrace{f'}_{\text{derivative of the outer}}(\underbrace{g(x)}_{\text{inner same}}) \cdot \underbrace{g'(x)}_{\text{derivative of the inner}}$$

or if $y = f(u)$ and $u = g(x)$, then

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx} = f'(u) \cdot g'(x) = f'(g(x)) \cdot g'(x).$$

Example 1: Find $F'(x)$ if $F(x) = (\underbrace{x^2}_{\text{inner}} - 1)^3$ outer

$$F'(x) = 3(\underbrace{x^2 - 1}_{\text{inner alone}})^2 \cdot (2x) = 6x(x^2 - 1)^2.$$

Example 2: Find the derivative of y if

a- $y = \dfrac{1}{\sqrt[4]{x^2+x+7}} = (\underbrace{x^2+x+7}_{\text{inner}})^{-\frac{1}{4}}$ ← outer

$$y' = -\frac{1}{4}(\underbrace{x^2+x+7}_{\text{inner alone}})^{-\frac{1}{4}-1} \cdot (\underbrace{2x+1}_{\text{Derivative of the inner}}) = -\frac{1}{4}(x^2+x+1)^{-\frac{5}{4}} \cdot (2x+1)$$

b- $y = \underset{\text{outer}}{\sqrt{\underbrace{x^3+x^2+1}_{\text{inner}}}} \longrightarrow y' = \dfrac{1}{2\sqrt{x^3+x^2+1}} \cdot (3x^2+2x) = \dfrac{3x^2+2x}{2\sqrt{x^2+x+1}}$

c- $y = \underset{\text{outer}}{\sin}(\underset{\text{inner}}{x^3}) \longrightarrow y' = \underset{\text{Derivative of outer}}{\cos}(\underset{\text{inner alone}}{x^3}) \cdot \underbrace{3x^2}_{\text{Derivative of inner}} = \cos x^3 \cdot 3x^2.$

d- $y = \sin^{\overset{\text{outer}}{3}} x \longrightarrow y' = 3(\sin x)^2 \cdot \cos x = 3\sin^2 x \cos x.$

$= (\sin x)^3$

e. $y = \left(\dfrac{x+1}{x-1}\right)^5 \longrightarrow y' = 5\left(\dfrac{x+1}{x-1}\right)^4 \cdot \dfrac{d}{dx}\left(\dfrac{x+1}{x-1}\right)$

$$= 5\left(\frac{x+1}{x-1}\right)^4 \cdot \frac{(x-1)(1)-(x+1)(1)}{(x-1)^2}$$

$$= 5\left(\frac{x+1}{x-1}\right)^4 \cdot \frac{x-1-x-1}{(x-1)^2} = 5\left(\frac{x+1}{x-1}\right)^4 \cdot \frac{-2}{(x-1)^2}$$

f- $y = \overset{\text{outer}}{\sqrt{\underbrace{x+\sqrt{x}}_{\text{inner}}}} \rightarrow y' = \dfrac{1}{2\sqrt{x+\sqrt{x}}} \cdot (x+\sqrt{x})' = \dfrac{1}{2\sqrt{x+\sqrt{x}}}\left(1+\dfrac{1}{2\sqrt{x}}\right).$

g- $y = \underbrace{\sin}_{\text{outer}}(\underbrace{\sin(\sin x)}_{\text{inner}})$

$$y' = \cos(\underbrace{\sin(\sin x)}_{\text{inner}}) \cdot \frac{d}{dx}(\overset{\text{outer}}{\sin}(\overset{\text{Inner}}{\underbrace{\sin x}_{\text{inner}}}))$$

$$= \cos(\sin(\sin(x))) \cdot \cos(\sin x) \cdot \cos x.$$

h- $y = x^2 \sec^2\left(\frac{1}{x}\right) = x^2 \left(\sec\frac{1}{x}\right)^2$

$$y' = 2x \sec^2\left(\frac{1}{x}\right) + x^2 \cdot 2\sec\left(\frac{1}{x}\right) \cdot \sec\frac{1}{x}\tan\frac{1}{x} \cdot \frac{-1}{x^2}$$

I- $y = e^{-5x} \longrightarrow y' = e^{-5x}(-5) = -5e^{-5x}$

J- $e^{4\sqrt{x}+x^2} \rightarrow y' = e^{4\sqrt{x}+x^2} \cdot (4\sqrt{x}+x^2)'$

$$= e^{4\sqrt{x}+x^2} \cdot \left(\frac{2}{\sqrt{x}} + 2x\right).$$

H- $y = \tan^{2}(\sin^3 t)$

$$= 2\left[\tan(\sin^3 t)\right] \cdot \left[\tan(\sin^3 t)\right]'$$

$$= 2\left[\tan(\sin^3 t)\right] \cdot \left[\sec^2(\sin^3 t)\right] \cdot \left[\sin^3 t\right]'$$

$$= 2\left[\tan(\sin^3 t)\right] \cdot \left[\sec^2(\sin^3 t)\right] \cdot \left[3\sin^2 t \cdot \cos t\right].$$

K- $y = e^{\sin t} + \sin(e^t)$

$y' = e^{\sin t} \cdot \cos t + \cos(e^t)\, e^t$.

<u>Example 3:</u> Find an equation of the tangent line to the curve at the given point

(a) $y = \sin(\sin x)$ at $(\pi, 0)$

<u>Solution:</u> we have a point, so we need to find the slope m which is the derivative at the given point.

$y' = \cos(\sin x)\,(\sin x)'$

$= \cos(\sin x) \cdot \cos x$

$y'(\pi) = \cos(\sin \pi) \cdot \cos \pi = \cos(0) \cdot \cos \pi = (1)(-1) = -1$

$y - y_1 = m(x - x_1)$

$y - 0 = -1(x - \pi)$

$\boxed{y = -x + \pi}$

(b) $y = \sin x + \sin^2 x$ at $(0, 0)$

<u>Exercise</u>