§3.7 - Implicit Differentiation

<u>Note</u>: So far, the functions we have met are given explicitly as expression in one variable, i.e.,

$$y = f(x)$$

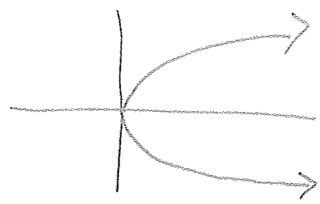
<u>However</u>, some functions are defined implicitly by a relation between x and y, for example

$$x^2 + y^2 = 9 \quad \text{or} \quad x^3 + y^3 - 6xy = 0 \quad \text{or} \quad \sin(x+y) = x^2 + y^2$$

Sometime, it is hard (or impossible) to write y alone as a function of x, in that case, we need implicit differentiation to find the derivative.

<u>Example 1</u>: (a) Find $\frac{dy}{dx}$ for $y^2 = x$.

(b) Find $\left.\frac{dy}{dx}\right|_{\substack{x=1 \\ y=1}}$ and $\left.\frac{dy}{dx}\right|_{\substack{x=1 \\ y=-1}}$



<u>Solution 1</u>:

write y alone, so

$$\sqrt{y^2} = \sqrt{x}$$

$$y = \sqrt{x} \quad \text{or} \quad y = -\sqrt{x}$$

$$\frac{dy}{dx} = \frac{1}{2\sqrt{x}} \quad \text{or} \quad \frac{dy}{dx} = -\frac{1}{2\sqrt{x}}$$

which one for which?

<u>Solution 2</u>: (Implicit Differentiation)

$$\frac{d}{dx}(y^2) = \frac{d}{dx}(x)$$

$$2y \cdot \frac{dy}{dx} = 1$$

$$\boxed{\frac{dy}{dx} = \frac{1}{2y}}$$

(b) $\left.\frac{dy}{dx}\right|_{(x,y)=(1,1)} = \frac{1}{2(1)} = \frac{1}{2}$.

$\left.\frac{dy}{dx}\right|_{(x,y)=(1,-1)} = \frac{1}{2(-1)} = \frac{1}{-2}$.

Exercise 1: Find $\frac{dy}{dx}$ for $xy=1$ by two ways.

Example 2: (a) Find $\frac{dy}{dx}$ for $x^2+y^2=9$

(b) Find the equation of the tangent line at $(1, \sqrt{8})$

Solution:

(a) $\frac{d}{dx}\left[x^2+y^2\right] = \frac{d}{dx}(9)$

$\frac{d}{dx}(x^2) + \frac{d}{dx}(y^2) = 0$

$2x + 2y \cdot \frac{dy}{dx} = 0 \longrightarrow 2y\frac{d}{dx} = -2x \rightarrow \frac{dy}{dx} = \frac{-x}{y} \quad (y \neq 0)$

(b) $m = \text{slope} = \left.\frac{dy}{dx}\right|_{(1,\sqrt{8})} = \frac{-1}{\sqrt{8}}$.

$y - y_1 = m(x - x_1)$

$y - \sqrt{8} = \frac{-1}{\sqrt{8}}(x-1) \longrightarrow \sqrt{8}y - 8 = -x + 1 \rightarrow \boxed{x + \sqrt{8}y = 9}$

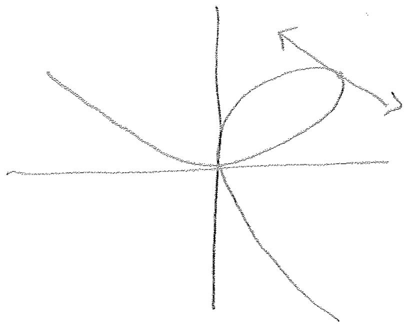
Exercise 2: Find $\frac{dy}{dx}$ for $x^3+y^3=1$

Example 3: (The Folium of Descartes)

(a) Find y' if $x^3+y^3=6xy$

(b) Find the tangent to the folium of Descartes at $(3,3)$.

(c) At what point in the first quadrant is the tangent line horizontal



Solution:

(a) $\frac{d}{dx}(x^3+y^3)=\frac{d}{dx}(6xy)$

$$3x^2+3y^2\cdot y' = 6xy'+6y$$

$$3y^2\cdot y'-6xy' = 6y-3x^2$$

$$(3y^2-6x)y' = 6y-3x^2 \longrightarrow \boxed{y'=\frac{6y-3x^2}{3y^2-6x}}$$

(b) $m=\text{slope}=y'_{(x,y)=(3,3)} = \frac{6(3)-3(3)^2}{3(3)^2-6(3)} = \frac{-9}{9} = -1.$

$$y-y_1=m(x-x_1) \longrightarrow y-3=-(x-3) \to y+x-6=0.$$

(c) $m=y'=0 \to \frac{6y-3x^2}{3y^2-6x}=0 \longrightarrow 6y=3x^2 \longrightarrow y=\frac{x^2}{2}$

$$x^3+y^3=6xy$$

$$x^3+\left(\frac{x^2}{2}\right)^3=6x\left(\frac{x^2}{2}\right) \longrightarrow x^3+\frac{x^6}{8}=3x^3 \to \frac{x^6}{8}=2x^3 \to x^6=16x^3.$$

$$x^3(x^3-16)=0 \to x=0 \quad \text{or} \quad x=\sqrt[3]{16}.$$

Example 4: Find y' if $xy = \cot(xy)$

Solution:

$$\frac{d}{dx}(xy) = \frac{d}{dx}(\cot(xy))$$

$$xy' + y = -\csc^2(xy)\cdot(xy' + y)$$

$$xy' + y = -\csc^2(xy)\cdot x \cdot y' - \csc^2(xy)\, y$$

$$xy' + \csc^2(xy)\cdot x y' = -\csc^2(xy)\cdot y - y$$

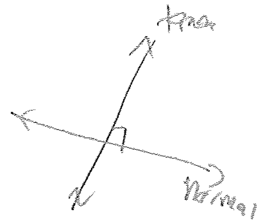
$$(x + \csc^2(xy)\cdot x)\, y' = -\csc^2(xy)\, y - y \rightarrow y' = \frac{-\csc^2(xy)\cdot y - y}{x + \csc^2(xy)\, x}.$$

Exercise 3: $\sin(xy) = \frac{1}{2}$.

Example 5: Find the tangent and normal lines to

(a) $x^2 + y^2 = 9$ at $(-1,3)$

Solution:



$$2xy^2 + 2x^2 y y' = 0 \rightarrow y' = \frac{-2xy^2}{2x^2y} = \frac{-y}{x}.$$

Slope of tangent line $= m = y'\Big|_{(x,y)=(-1,3)} = \frac{-3}{-1} = 3.$

$$y - y_1 = m(x - x_1)$$

$$y - 3 = 3(x+1).$$

Slope of normal line $= \frac{-1}{m} = -\frac{1}{3}$, so $y - y_1 = m(x - x_1)$

$$y - 3 = -\frac{1}{3}(x+1).$$

(b) $x^2\cos^2 y - \sin y = 0$ at $(0, \pi)$

Solution: Differentiate both sides with respect to x

$$\frac{d}{dx}\left(x^2\cos^2 y - \sin y\right) = \frac{d}{dx}(0)$$

$$2x\cos^2 y - 2x^2\cos y \cdot \sin y \cdot y' - \cos y \cdot y' = 0$$

$$\boxed{y' = \frac{2x\cos^2 y}{2x^2\cos y\sin y + \cos y}}$$

$$\text{Slope} = m = y'\Big|_{(0,\pi)} = \frac{0}{\quad} = 0$$

So No normal line! (why).