§3.8 - Derivative of Inverse Function and Logarithms

- Inverse Functions.
- Logarithms.
- Derivative of inverse function
- Logarithmic Differentiation.

1- Inverse functions (Pre-Calculus)

Definition:

Let f be a function. An inverse function is another function f^{-1} such that

$$\underbrace{f}_{\text{outer}}\Big(\underbrace{f^{-1}(x)}_{\text{inner}}\Big) = x \quad \text{and} \quad f^{-1}(f(x)) = x.$$

where $f(f^{-1}(x)) = (f \circ f^{-1})(x)$.

(The function and the inverse are cancelling each other).

Example 1: (a) $f(x) = x+5$, then $f^{-1}(x) = x-5$ (we will see how to find it shortly)

- $f(f^{-1}(x)) = f(x-5) = x-5+5 = x.$
- $f^{-1}(f(x)) = f^{-1}(x+5) = x+5-5 = x.$

(b) $f(x) = x^2$ $(x > 0)$, then $f^{-1}(x) = \sqrt{x}$ because

- $f(f^{-1}(x)) = f(\sqrt{x}) = (\sqrt{x})^2 = x$
- $f^{-1}(f(x)) = f^{-1}(x^2) = \sqrt{x^2} = |x| = x$ (since $x > 0$)

Question: Is every function has an inverse? how do we tell if the function has an inverse?

Answer: No! we use the horizontal line test if we have the graph of the function!

To find the inverse

algebraically

Step 1: write $y = f(x)$

Step 2: Switch x with y to get $x = f(y)$

Step 3: Solve for y, i.e., isolate y alone to get

$$y = f^{-1}(x)$$

Geometrically

reflect the graph of $f(x)$ about the line $y = x$ to get the graph of $f^{-1}(x)$.

Example 1: Find the inverse of $f(x) = x + 7$

Step 1: $y = x + 7$

Step 2: $x = y + 7$

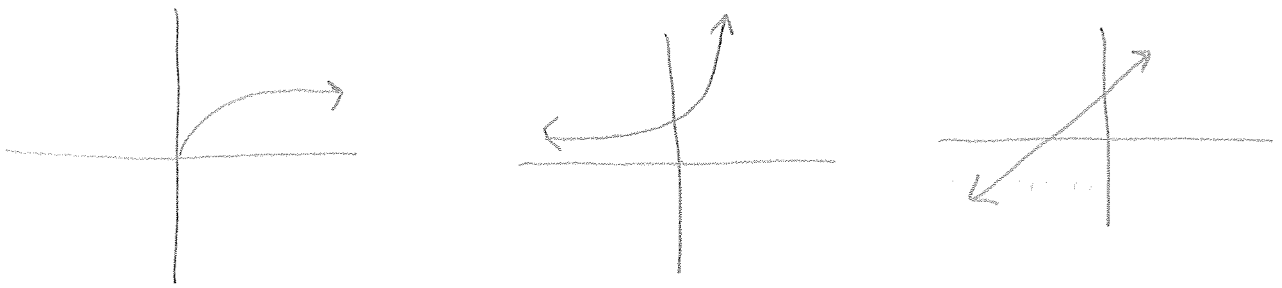
Step 3: $x - 7 = y \rightarrow \boxed{f^{-1}(x) = x - 7}$

Exercise 1: Find the inverse of

(1) $f(x) = 3x+2$ (2) $f(x) = x^2-1$, $x>0$

(3) $f(x) = \frac{1}{x}$ (4) $f(x) = \sqrt{x}$

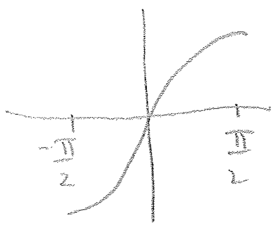
Example 2: Find the graph of the inverse function.



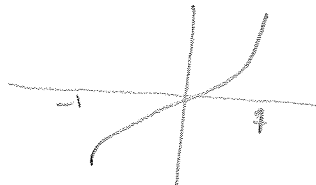
Note: The domain of $f^{-1}(x)$ is the range of $f(x)$.

Example 3: (Inverse of trigonometric functions)

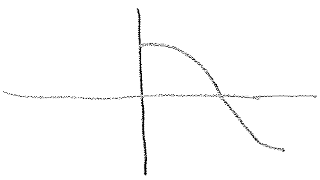
(1) $y = \sin x$, $x \in [-\frac{\pi}{2}, \frac{\pi}{2}]$ has an inverse $f^{-1}(x) = \sin^{-1} x$.



domain $[-1,1]$ Range $[-\frac{\pi}{2}, \frac{\pi}{2}]$.



(2) $f(x) = \cos x$, $x \in [0, \pi]$ has an inverse $f^{-1}(x) = \cos^{-1} x$



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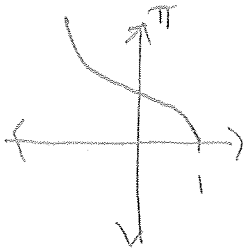
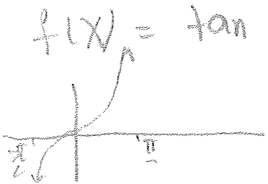
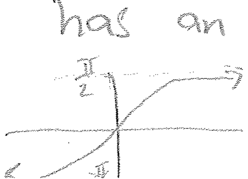


domain $[-1, 1]$ Range $[0, \pi]$

(3) $f(x) = \tan x$, $x \in [-\frac{\pi}{2}, \frac{\pi}{2}]$ has an inverse $f^{-1}(x) = \tan^{-1} x$



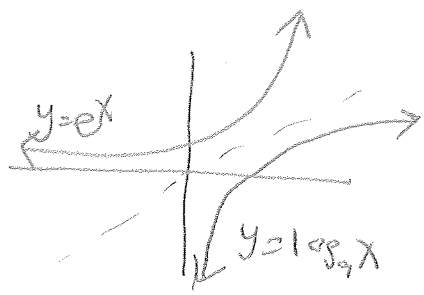


Domain $(-\infty, \infty)$ Range $[-\frac{\pi}{2}, \frac{\pi}{2}]$

2- The logarithm

Consider $f(x) = a^x$, $a>0$ ... exponential function.

<u>Question</u>: Does $y=a^x$ has an inverse?



<u>Answer</u>: Yes, by the horizontal line test.

<u>Definition</u>:

$$f^{-1}(x) = \log_a x$$

<u>Note</u>:

- $f^{-1}(f(x)) = \log_a a^x = x$
- $f(f^{-1}(x)) = a^{\log_a x} = x$.

So we have

$$\boxed{y = \log_a x \iff a^y = a^{\log_a x} \qquad a^y = x.}$$ — ☆

<u>Special case</u>: If $a=e$, then $\log_a = \ln$ ← "ell n"

<u>Properties of logarithms</u> (can be proved by ☆)

(1) $\log_a x + \log_a y = \log_a xy$

(2) $\log_a x - \log_a y = \log_a \frac{x}{y}$

(3) $\log_a x^n = n \log_a x$

(4) $\log_a a = 1$

(5) $\log_a 1 = 0$

(6) $\log_a x = \frac{\log_b x}{\log_b a}$ ... change of base.

3- Derivative of inverse function

Goal: we want to find $\frac{d}{dx}\left(f^{-1}(x)\right)$. without

Strategy: write $y = f^{-1}(x)$, we want y'.

$$f(y) = f(f^{-1}(x))$$

$$f(y) = 1$$

$$f'(y) \cdot y' = 1 \rightarrow \quad y' = \frac{1}{f'(y)} = \frac{1}{f'(f^{-1}(x))}$$

(The slope of f^{-1} is reciprocal to the slope of f).

Example 1: Let $f(x) = x^3 - 3x^2 - 1$. Find $\frac{d}{dx}(f^{-1})$ at $(3, -1)$.

Solution:

$$y = f^{-1}(x) \longrightarrow \quad f(y) = x$$

$$f'(y) \cdot y' = 1 \rightarrow y' = \frac{1}{f'(y)} = \frac{1}{3y^2 - 6y}$$

$$y'\Big|_{\substack{x=3 \\ y=-1}} = \frac{1}{3(-1)^2 - 6(-1)} = \frac{1}{9}$$

Example 2: Find $\frac{d}{dx}(\ln x)$?

let $y = \ln x$

$$e^y = e^{\ln x} \rightarrow e^y = x \rightarrow e^y \cdot y' = 1 \rightarrow y' = \frac{1}{e^y} = \frac{1}{x}$$

So $$\frac{d}{dx}(\ln x) = \frac{1}{x} \quad \left(\text{in general, } \frac{d}{dx}\ln|u| = \frac{1}{|u|} \cdot \frac{du}{dx} \right).$$

Exercise 1: Find y' if $y = \log_a x$.

Example 2: Find y' for

(1) $y = \ln(x^2) \rightarrow y' = \frac{1}{x^2} \cdot 2x = \frac{2x}{x^2} = \frac{2}{x}$.

(2) $y = \ln(2x+3) \rightarrow y' = \frac{1}{2x+3} \cdot (2) = \frac{2}{2x+3}$.

(3) $y = x \ln x \rightarrow y' = x \cdot \frac{1}{x} + 1 \cdot \ln x = 1 + \ln x$

(4) $y = \ln(\ln x) \rightarrow y' = \frac{1}{\ln x} \cdot \frac{1}{x} = \frac{1}{x \ln x}$.

(5) $y = \sec(\ln \theta) \rightarrow y' = \sec(\ln\theta) \tan(\ln\theta) \cdot \frac{1}{\theta}$.

4- Logarithmic Differentiation

we take ln of both sides and then we differentiate.

Example 1: Find y' if

$$y = \frac{(x^2+1)(x-2)^{5/2}}{(x+7)}, \quad x > 5$$

Solution:

$$\ln y = \ln \frac{(x^2+1)(x-2)^{\frac{5}{2}}}{(x+7)} = \ln (x^2+1)(x-2)^{\frac{5}{2}} - \ln(x+7)$$

$$\ln y = \ln(x^2+1) + \ln(x-2)^{5/2} - \ln(x+7)$$

$$\ln y = \ln(x^2+1 + \frac{5}{2}\ln(x-2) - \ln(x+7)$$

Now we take the derivative of both sides

$$\frac{1}{y}\cdot y' = \frac{1}{x^2+1}(2x) + \frac{5}{2}\cdot\frac{1}{x-2}(1) - \frac{1}{x+7}(1)$$

$$y' = y\left[\frac{2x}{x^2+1} + \frac{5}{2(x-2)} - \frac{1}{x+7}\right]$$

$$y' = \frac{(x^2+1)(x-2)^{5/2}}{(x+7)}\left[\frac{2x}{x^2+1} + \frac{5}{2(x-2)} - \frac{1}{x+7}\right]$$

Example : Find y' for $y = \dfrac{\theta\sin\theta}{\sqrt{\sec\theta}}$

$$\ln y = \ln\frac{\theta\sin\theta}{\sqrt{\sec\theta}} = \ln\theta + \ln\sin\theta - \frac{1}{2}\ln\sec\theta$$

$$\frac{1}{y}\cdot y' = \frac{1}{\theta} + \frac{1}{\sin\theta}\cdot\cos\theta - \frac{1}{2}\cdot\frac{1}{\sec\theta}\cdot\sec\theta\tan\theta$$

$$\frac{1}{y}y' = \frac{1}{\theta} + \cot\theta - \frac{\tan\theta}{2} \rightarrow y' = \frac{\theta\sin\theta}{\sqrt{\sec\theta}}\left[\frac{1}{\theta} + \cot\theta - \frac{\tan 2\theta}{2}\right]$$

Example :

Recall:

$$\frac{d}{dx}(x^n) = n x^{n-1}$$

$$\frac{d}{dx}(a^x) = a^x \ln a$$

variable

$$\frac{d}{dx}(x^x) = ?$$

variable?

Set $y = x^x$

$$\ln y = \ln x^x = x \ln x$$

Differentiate,

$$\frac{1}{y} \cdot y' = x \cdot \frac{1}{x} + \ln x$$

$$\frac{1}{y} \cdot y' = 1 + \ln x \rightarrow y' = y(1 + \ln x)$$

$$\boxed{y' = x^x + x^x \ln x}$$

Exercise : Find y' if

(1) $y = (\sqrt{x})^x$

(3) $y = x^{\ln x}$

(2) $y = (\ln x)^x$

(4) $y = (\ln x)^x$.