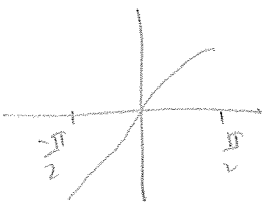
§ 3.9 Derivative of the inverse Trigonometric Functions

1- Derivative of the basic inverse Trigonometric function

2- Derivative of functions that involve inverse trigonometric Functions.

1- Derivative of the basic inverse trigonometric functions

Recall: $f(x) = \sin x \ , \ x \in [-\frac{\pi}{2}, \frac{\pi}{2}]$ has an inverse



$$f^{-1}(x) = \sin^{-1} x \quad (= \arcsin x).$$

domain $[-1, 1]$ Range $[-\frac{\pi}{2}, \frac{\pi}{2}]$ with $\sin(\sin^{-1} x) = x$

$\sin^{-1}(\sin x) = x$.

Similarly, for the rest.

Example 1: Find $\frac{d}{dx}(\sin^{-1} x)$.

Solution:

let $y = \sin^{-1} x$, Find y'.

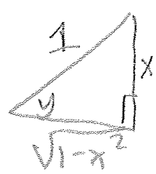
$\sin y = \sin(\sin^{-1} x))$

$\sin y = x$

Differentiate both sides with respect to x

$\cos y \cdot y' = 1 \leadsto y' = \frac{1}{\cos y}$

Now $\sin y = x = \frac{x}{1} = \frac{opp}{hypo}$



Recall: Pythagorian Theorem

$(hyp)^2 = (adj)^2 + (opp)^2$

$1 = (adj)^2 + x^2$

$(adj)^2 = 1 - x^2$

$adj = \sqrt{1-x^2}$

So $\cos y = \frac{adj}{hyp} = \frac{\sqrt{1-x^2}}{1} = \sqrt{x^2-1}$

So $y' = \dfrac{1}{\cos y} = \dfrac{1}{\sqrt{1-x^2}} \rightsquigarrow \dfrac{d}{dx}(\sin^{-1} x) = \dfrac{1}{\sqrt{1-x^2}}$.

<u>Exercise 2:</u> Find $\dfrac{d}{dx}(\cos^{-1} x)$.

<u>Example 2:</u> Find $\dfrac{d}{dx}(\tan^{-1} x)$.

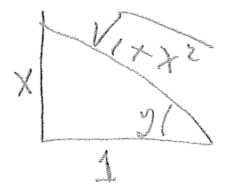
let $y = \tan^{-1} x$

$\tan y = x$

Differentiate with respect to x

$\sec^2 y \cdot y' = 1 \to y' = \dfrac{1}{\sec^2 y}$

Now $\tan y = \dfrac{x}{1} = \dfrac{\text{opp}}{\text{adj}}$



By Pythagorean's Theorem,

$(\text{adj})^2 + (\text{opp})^2 = (\text{hypo})^2$

$1^2 + x^2 = (\text{hypo})^2$

$\text{hypo} = \sqrt{1+x^2}$

Now $\sec y = \dfrac{\text{hypo}}{\text{adj}} = \dfrac{\sqrt{1+x^2}}{1} = \sqrt{1+x^2}$

So $y' = \dfrac{1}{\sec^2 y} = \dfrac{1}{1+x^2} \rightsquigarrow \dfrac{d}{dx}(\tan^{-1} x) = \dfrac{1}{1+x^2}$.

<u>Exercise 2:</u> Find $\dfrac{d}{dx}(\cot^{-1} x)$

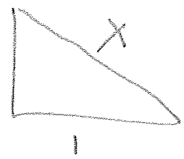
<u>Example 3:</u> Find $\dfrac{d}{dx}(\sec^{-1} x)$.

$y = \sec^{-1} x \rightsquigarrow \sec y = x$

$\sec y \tan y \cdot y' = 1$

$y' = \dfrac{1}{\sec y \tan y} = \dfrac{1}{x\sqrt{x^2-1}}$

$\sec y = \dfrac{\text{hypo}}{\text{adj}} = \dfrac{x}{1}$



So $\text{opp} = \sqrt{x^2-1}$

$\tan y = \dfrac{\text{opp}}{\text{adj}} = \sqrt{x^2-1}$

Exercise 3: Find $\frac{d}{dx}(\csc^{-1}x)$?

Summary

1. $\frac{d}{dx}(\sin^{-1}u) = \frac{1}{\sqrt{1-u^2}} \cdot \frac{du}{dx}$

2. $\frac{d}{dx}\cos^{-1}u = \frac{-1}{\sqrt{1-u^2}} \cdot \frac{du}{dx}$

3. $\frac{d}{dx}(\tan^{-1}u) = \frac{1}{1+u^2} \cdot \frac{du}{dx}$

4. $\frac{d}{dx}(\cot^{-1}u) = \frac{-1}{1+u^2} \cdot \frac{du}{dx}$

5. $\frac{d}{dx}(\sec^{-1}u) = \frac{1}{u\sqrt{u^2-1}} \cdot \frac{du}{dx}$

6. $\frac{d}{dx}(\csc^{-1}u) = \frac{-1}{u\sqrt{u^2-1}} \cdot \frac{du}{dx}$

2- Derivative of functions that involve inverse trigonometric functions

Example 4: Find the derivative of the following functions

1. $y = \tan^{-1}(\sqrt{x})$

$$y' = \frac{1}{1+(\sqrt{x})^2} \cdot \frac{d}{dx}(\sqrt{x})$$

$$= \frac{1}{1+x} \cdot \frac{1}{2\sqrt{x}}$$

2. $y = \sqrt{\tan^{-1}x}$

$$y' = \frac{1}{2\sqrt{\tan^{-1}x}} \cdot \frac{d}{dx}(\tan^{-1}x)$$

$$= \frac{1}{2\sqrt{\tan^{-1}x}} \cdot \frac{1}{1+x^2}$$

3. $y = \cos^{-1}(e^x)$

$$y' = \frac{-1}{\sqrt{1-(e^x)^2}} \cdot e^x$$

4- $y = \arcsin(3-2x)$

$$y' = \frac{1}{\sqrt{1-(3-2x)^2}} \cdot (-2)$$

5. $y = \sin^{-1}(\sqrt{\sin x})$

$$y' = \frac{1}{\sqrt{1-(\sqrt{\sin x})^2}} \cdot \frac{d}{dx}(\sqrt{\sin x})$$

$$= \frac{1}{\sqrt{1-\sin x}} \cdot \frac{1}{2\sqrt{\sin x}} \cdot \cos x$$

7. $\tan^{-1}(xy) = x + x^2 y$

$$\frac{1}{1+(xy)^2} \cdot (y + xy') = 1 + 2xy + x^2 y'$$

$\vdots$

$y' = \cdots$

6. $y = \tan^{-1}x + \tan^{-1}\left(\frac{1}{x}\right)$

$$y' = \frac{1}{1+x^2} + \frac{1}{1+\left(\frac{1}{x}\right)^2} \cdot \left(-\frac{1}{x^2}\right)$$

$$y' = \frac{1}{1+x^2} + \frac{\cancel{x^2}}{x^2+1}\left(-\frac{1}{\cancel{x^2}}\right)$$

$$y' = \frac{1}{1+x^2} - \frac{1}{1+x^2} = 0$$

So y is constant function

$\tan^{-1}x + \tan^{-1}\left(\frac{1}{x}\right) = c$, Put $x = 1$

$\frac{\pi}{4} + \frac{\pi}{4} = c \sim c = \frac{\pi}{2}$

So we get $\tan^{-1}x + \tan^{-1}\left(\frac{1}{x}\right) = \frac{\pi}{2}$.

Exercise 4: Prove $\sin^{-1}x + \cos^{-1}x = \frac{\pi}{2}$ using Example 4.6.