# Part III : Applications of Differentiation

## §4.1 - Maximum and Minimum Values

One of the most important applications of differential calculus are optimization problem (Find the optimal (best) way to do something) and in most cases, these optimization problems are reduced to finding the minimum and maximum of a function.

### 1- Minimum and maximum of a function

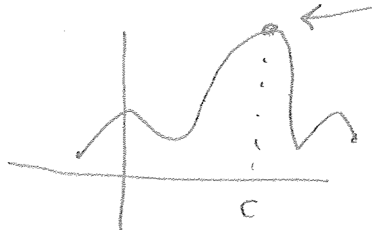
Absolute Maximum (Global maximum)

**Algebra**

$f(c)$ is an absolute maximum if

$$f(x) \leq f(c) \quad , \forall x \in Domain(f)$$

**Geometry**




absolute maximum (only one, the maximum of all)

**Exercise 1:** Define the absolute minimum of a function.

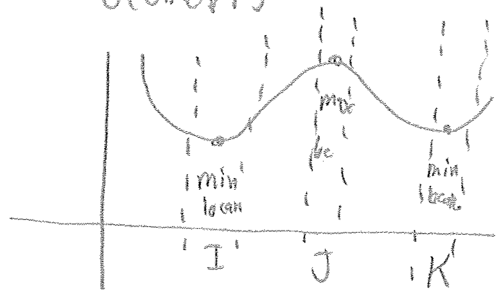
local maximum

**Algebra**

$f(c)$ is a local maximum if

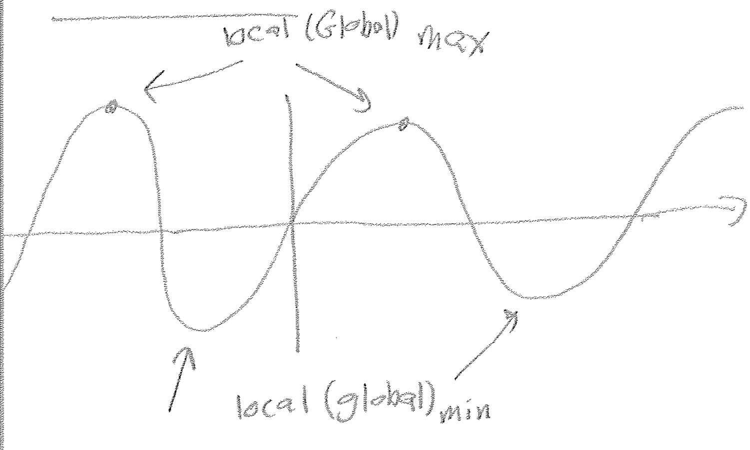
$f(x) \leq f(c)$ near $c$.

**Geometry**

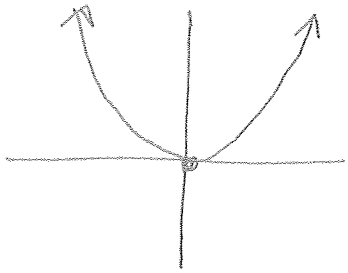


Exercise 2: Define local minimum value of $f(x)$.

Example 1: Consider the function $f(x) = \sin x$.

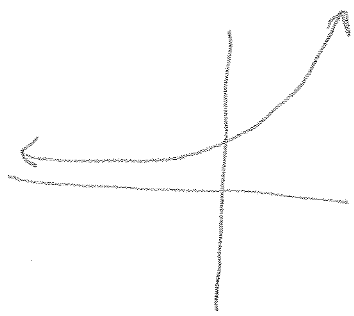



Example 2: $f(x) = x^2$



- It has global min at $(0,0)$ (local min)
- It has No global (local) maximum

Example 3: $f(x) = e^x$



- It has neither global maximum neither global minimum.

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Question: When can we gaurantee that we would have global max or global min?

Answer: [The Extreme Value Theorem]

If $f$ is continuous on closed interval $[a,b]$, then it has both a global maximum and a global minimum in $[a,b]$

Next, the extreme value theorem guarantees us that there is a global max or min. Now we want to find them.

<u>Recall</u>: If $(c, f(c))$ is a local maximum, then the slope of the tangent line is zero, i.e., $f'(c) = 0$!

<u>Definition</u>: A number $c$ is called a <u>critical point of $f$</u> if $f'(c) = 0$ <u>or</u> $f'(c)$ does not exist.

<u>Example 4</u>: Find the critical points of the following.

(a) $y = x^3 + x^2 - x$

$f'(x) = 3x^2 + 2x - 1$

$f'(x) = 0$

$3x^2 + 2x - 1 = 0$

$\boxed{x = 1}$ or $\boxed{x = -3}$

(b) $g(x) = \sqrt{1 - x^2}$

$g'(x) = \dfrac{-2x}{2\sqrt{1-x^2}} = \dfrac{-x}{\sqrt{1-x^2}}$

$g'(x) = 0 \rightarrow \dfrac{-x}{\sqrt{1-x^2}} = 0$

$-x = 0 \rightarrow \boxed{x = 0}$

$g'(x)$ DNE if

$1 - x^2 = 0 \rightarrow x = \pm 1$

(c)

$f(x) = \dfrac{x - 1}{x^2 - x + 1}$

$f'(x) = \dfrac{(x^2 - x + 1) - (x-1)(2x-1)}{(x^2 - x + 1)^2}$

$= \dfrac{\cancel{x^2} - x + 1 - \cancel{2x^2} + 3x - 1}{(x^2 - x + 1)^2}$

$= \dfrac{-x^2 + 2x}{(x^2 - x + 1)^2}$

$f'(x) = 0$

$x(-x + 2) = 0$

$\boxed{x = 0}$ or $\boxed{x = 2}$

$f'(x)$ DNE if

$x^2 - x + 1 = 0$

No solution

To find the absolute max or min on closed interval

1- Find the critical points $c$ and find $f(c)$.

2- Find $f(a)$ & $f(b)$.

3- global max (or global min) is the one that is largest (smallest)

<u>Example 5:</u> Find the global maximum or global minimum.

for $f(x) = x^3 - 3x + 1$ on $[0, 3]$

<u>Solution:</u>

1- we find the cricital points

$f'(x) = 3x^2 - 3$.

$f'(x) = 0$

$3x^2 - 3 = 0$

$x = 1$ or $x = -1$

$f'(x)$ DNE

No solution.

Candidate

| $c$ | $f(c)$ | |
|---|---|---|
| $a = 0$ | 1 | |
| $b = 3$ | 19 | ← global max |
| 1 | -1 | ← global min. |
| -1 | 3 | |

2- Find $f(a) = f(0) = 1$

$f(b) = f(3) = 19$

Exercise 3: $f(x) = 3x^4 - 4x^3 - 12x^2 + 1$ on $[-2, 3]$

Example 6: $f(t) = t\sqrt{4-t^2}$ on $[-1, 2]$

Solution:

1- $f'(t) = \dfrac{t(-2t)}{2\sqrt{4-t^2}} + \sqrt{4-t^2}$

$= \dfrac{-t^2}{\sqrt{4-t^2}} + \sqrt{4-t^2} = \dfrac{-t^2+4-t^2}{\sqrt{4-t^2}} =$

$= \dfrac{4-2t^2}{\sqrt{4-t^2}}$

| $c$ | $f(c)$ | |
|---|---|---|
| $-1$ | $-\sqrt{3}$ | ← global min |
| $2$ | $0$ | |
| $\sqrt{2}$ | $2$ | ← global max |
| ~~$-\sqrt{2}$~~ | ~~$-2$~~ | ~~← global min~~ |
| ~~$2$~~ | ~~$0$~~ | |
| ~~$-2$~~ | ~~$0$~~ | |

$f'(t) = 0$

$4 - 2t^2 = 0$

$t = \sqrt{2}$ or $t = -\sqrt{2}$

$f'(t)$ DNE if

$4 - t^2 = 0$

$t = 2$ or $t = -2$

Exercise 4: $f(x) = x + \dfrac{1}{x}$, $[0.2, 4]$.

Example 7: $f(x) = xe^{-x}$ on $[-1, 1]$

Solution:

$f'(x) = e^{-x} - xe^{-x} = e^{-x}(1-x)$

| $c$ | $f(c)$ | |
|---|---|---|
| $-1$ | $-e^{1}$ | ← global min |
| $1$ | $e^{-1}$ | ← global max |

$f'(x) = 0$

$1 - x = 0$

$x = 1$

$f'(x)$ DNE

None

Exercise 5: $f(x) = \cos x + \sin x$ on $[-\frac{\pi}{2}, \frac{\pi}{4}]$.