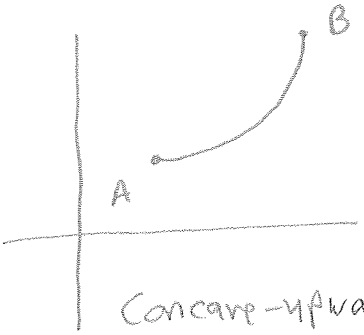
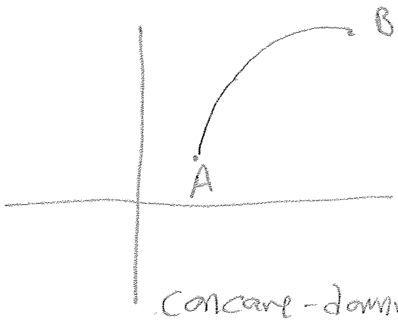
§ 4.4- Concavity and Curve Sketching

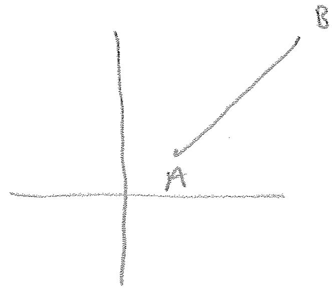
Increasing function has three cases



Concave-upward
(CU)



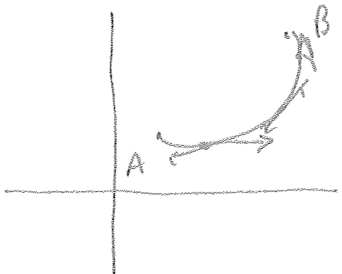
Concave-downward
CD



~~No~~ No Curvature!

<u>Question</u> : How to distinguish between these three types of Behaviour?

<u>Answer</u> :- Recall :- If $g(x)$ is increasing, then $g'(x) > 0$.



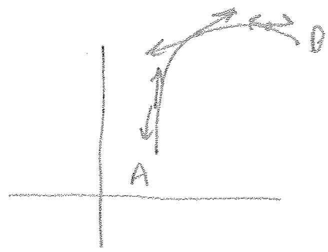
Slope increasing

$\Downarrow$

$f'(x)$ increasing

$\Downarrow$

$f''(x) > 0$.



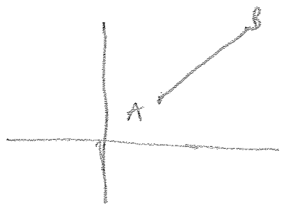
Slope is decreasing

$\Downarrow$

$f'(x)$ is decreasing

$\Downarrow$

$f''(x) < 0$



Slope is consta

$\Downarrow$

$f'(x) = c$

$\Downarrow$

$f''(x) = 0$, for all x.

<u>Concavity Test</u>

1- If $f''(x) > 0$, then the curve is CU.

2- If $f''(x) < 0$, then the curve is CD.

3- If $f''(x) = 0$ (for all x), then the curve has no curvature (line)

Example 2: Find the intervals where the following function increases, decreases, has local max/min and CU or CD.

(a) $f(x) = 4x^3 + 3x^2 - 6x + 1$

$f'(x) = 12x^2 + 6x - 6$

$f''(x) = 24x + 6$

$f'(x) = 0 \rightarrow 12x^2 + 6x - 6 = 0$

$x = -1$ or $x = +\frac{1}{2}$

-1 $\quad$ $+\frac{1}{2}$

++++ $\quad$ - - - $\quad$ ++++

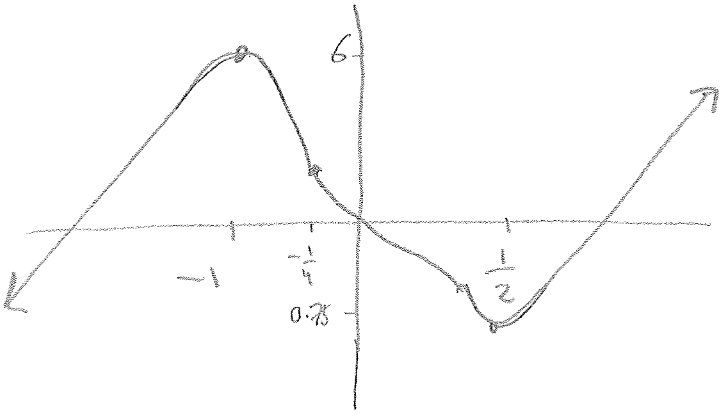
local max $\quad$ local min

$f''(x) = 0 \rightarrow 24x + 6 = 0$

$x = -\frac{1}{4}$

- - - - $\quad -\frac{1}{4} \quad$ ++++

CD $\quad$ CU



(b) (exercise) $y = (-2)x^3 + 6x^2 - 3$

(c) $y = x - \sin x \quad , x \in [0, 4\pi]$.

$f'(x) = 1 - \cos x$

$f'(x) = 0 \rightarrow 1 - \cos x = 0$

$\cos x = 1$

$x = 0, 2\pi, 4\pi$

0 + + + + 2π + + + + 4π

increasing

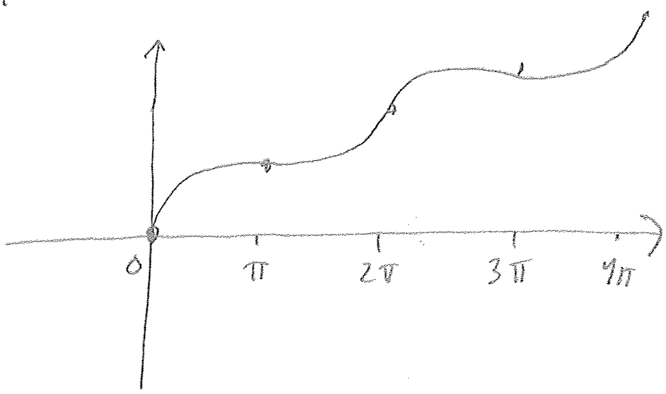
$f''(x) = -\sin x$

$f''(x) = 0 \rightarrow \sin x = 0$

$x = 0, \pi, 2\pi, 3\pi, 4\pi$

0 — — — π + + + 2π — — — 3π + + + + 4π

CD CU CD CU



(d) (Exercise) $y = \cos x + \sin x \quad , 0 \le x \le 2\pi$.

Curve Sketching

1- Find the domain of $f(x)$.

2- Find the intercept by solving $y=0$ (if possible) and set $x=0$ to find $y=f(0)$.

3- Find the horizontal and vertical asymptote.

4. Find $f'(x)$ and $f''(x)$.

5. Find the local max and local min

6. Find the interval where the function is increasing/decreasing.

7. Find the concavity of the curve.

Example 1: Sketch $y=f(x)=x^5-5x^4$

1. Domain of $f(x)$ is $(-\infty,\infty)$ since we don't have any value that makes $f(x)$ undefined.

2- $y=0 \to x^5-5x^4=0 \to x^4(x-5)=0 \to x=0$ or $\cancel{y}\,x=5$

So x-intercepts are $(0,0)$ and $(5,0)$.

Now set $x=0 \to y=0 \to$ so y-intercept is $(0,0)$.

3- horizontal asymptote

$\lim_{x\to\pm\infty} f(x) = \pm\infty$

No horizontal asymptote

Vertical asymptote.

None.

4- $f'(x) = 5x^4 - 20x^3$

5- $f'(x) = 0$

$5x^4 - 20x^3 = 0$

$5x^3(x-4) = 0$

$x = 0$ or $x = 4$

6-

		0		4	
sign of $f'(x)$	$+++$		$----$		$+++$

local max (at 0)

local min (at 4)

$f''(x) = 20x^3 - 60x^2$

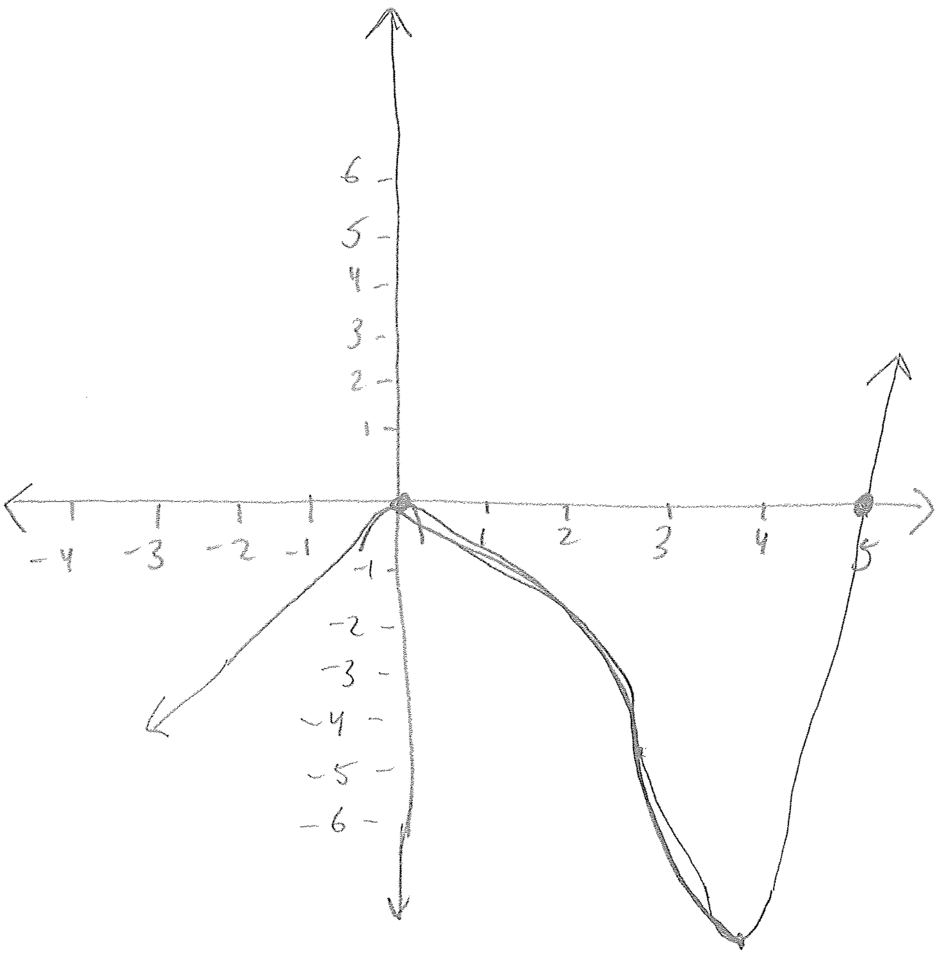
$f''(x) = 0$

$20x^3 - 60x^2 = 0$

$20x^2(x-3) = 0$

$x = 0$ or $x = 3$

		0		3	
sign of $f''(x)$	$----$		$----$		$+++$
	CD		CD		CU



Example 2: Sketch $y = f(x) = \dfrac{x}{x-2}$.

1- Domain of f is all the points except $x = 2$.

2- $f(x) = 0 \to \dfrac{x}{x-2} = 0 \to \boxed{x = 0}$, so x-intercept is $(0,0)$.

y-intercept is $(0,0)$.

3- $\lim\limits_{x\to\infty} f(x) = \lim\limits_{x\to\infty} \dfrac{x}{x-2} = 1$, so $y = 1$ is horizontal asymtote.

and $x = 1$ is vertical asymptote.

4- $f'(x) = \dfrac{-2}{(x-2)^2}$

$f''(x) = 0$ No solution

$f'(x)$ DNE at $x = 2$

5-

6- $- - - - \; 2 \; - - - -$

No max nor min

$f''(x) = \dfrac{4}{(x-2)^3}$.

$f''(x) = 0$ No solution, $f''(x)$ DNE at $x = 2$

7. $- - - \; 2 \; + + +$

CD CU