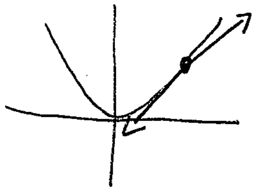
Part II – Integration

Calculus

Derivative

- It is used to find the <u>slope</u> of the tangent line to a curve.



- Definition

$$f'(a) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}$$

$$= \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$$

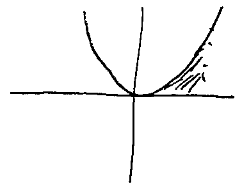
- Rules From definition
  - Sum rule
  - Product rule
  - Quotient rule
  - Chain rule
  - ⋮

- From the definition, Derivative of the famous functions.

$\Longrightarrow$ We want to do the same thing here

Integral

- It is used to compute the area under the curve.



- Definition

  ....

- Rules from the definition

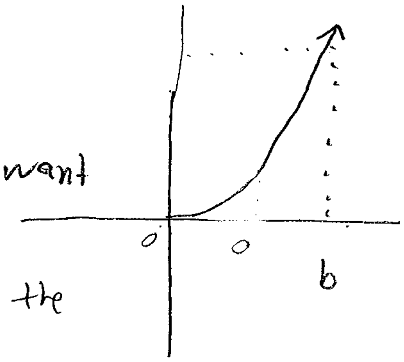
  ⋮

- from the definition

  ⋮

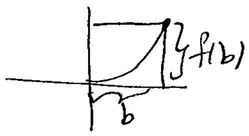
Definition of the integral

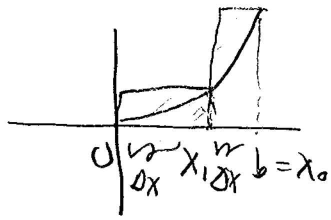
Remember, the integral is used to find the area ~~of~~ under a curve over an interval $[a,b]$.

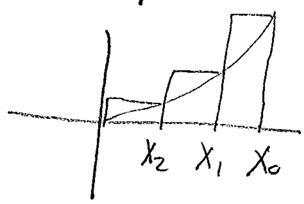


Idea we cover the area that we want to compute using rectangles (the highest the number of rectangles, the better the estimation)

n = # of rectangle	Area of the rectangle
1	$\underbrace{b}_{\text{width}} \times f(b)$
2	$\Delta x\, f(x_1) + \Delta x\, f(x_0)$
3	$\Delta x\, f(x_2) + \Delta x\, f(x_1) + \Delta x\, f(x_0)$
$\vdots$	
n	$\underbrace{\Delta x\, f(x_n) + \Delta x\, f(x_{n-1}) + \cdots + \Delta x\, f(x_0)}_{\text{Riemann-Sum}}$







So $\text{Area} = \sum_{k=0}^{n} \Delta x\, f(x_k)$, where $x_k = a + k\Delta x$, $\Delta x = \frac{b-a}{n}$

$$= \frac{(b-a)}{n} \sum_{k=0}^{n} f(x_n)$$

and so $\boxed{\int_a^b f(x)\,dx = \lim_{n\to\infty} \frac{b-a}{n} \sum_{k=0}^{n} f(x_k)}$

Note:

$$\int_a^b f(x)\,dx = \int_a^b f(t)\,dt$$

upper integral limit → b; integral sign → $\int$; $f(x)$ → integrand; lower integral limit → a

- You can choose any sample points to calculate the integral, the answer will <u>not</u> change.

<u>Disadvantage of the definition</u>

It is so <u>difficult</u> to compute the integral of any function

<u>Example 1:</u> Find $\int_0^1 x\,dx$ using the definition.

<u>Solution:</u> Note $f(x) = x$

$$\int_0^1 x\,dx = \lim_{n\to\infty} \Delta x \sum_{K=0}^{n} f(x_K), \qquad \Delta x = \frac{b-a}{n} = \frac{1-0}{n} = \frac{1}{n}$$

$$x_K = a + K\Delta x = 0 + K\cdot\frac{1}{n} = \frac{K}{n}.$$

$$= \lim_{n\to\infty} \frac{1}{n}\cdot\sum_{K=0}^{n} f\left(\frac{K}{n}\right)$$

$$= \lim_{n\to\infty} \frac{1}{n}\sum_{K=0}^{n}\frac{K}{n} = \lim_{n\to\infty}\frac{1}{n}\cdot\frac{1}{n}\sum_{K=0}^{n} K = \lim_{n\to\infty}\frac{1}{n^2}\sum_{K=0}^{n} K$$

$$= \lim_{n\to\infty}\frac{1}{n^2}\cdot\frac{n(n+1)}{2} = \lim_{n\to\infty}\frac{n^2+n}{2n^2} = \lim_{n\to\infty}\frac{n^2(1+\frac{1}{n})}{2n^2} = \lim_{n\to\infty}\frac{1+\frac{1}{n}}{2}$$

$$= \frac{1}{2}.$$

<u>Exercise 1:</u> Using the definition, compute $\int_0^1 x^2\,dx$ $\left(\text{Hint } \sum_{K=0}^{n} K^2 = \frac{n(n+1)(2n+1)}{6}\right)$

- Properties of the integral (from the definition)

$$\int_a^b f(x)\,dx = \lim_{n\to\infty} \Delta x \sum_{k=0}^{n} f(x_k), \qquad \Delta x = \frac{b-a}{n}, \quad x_k = a + k\Delta x$$

(1) $$\int_a^b f(x)+g(x)\,dx = \int_a^b f(x)\,dx + \int_a^b g(x)\,dx$$

$$\int_a^b [f(x)+g(x)]\,dx = \lim_{n\to\infty} \Delta x \sum_{k=0}^{n}\left[f(x_k)+g(x_k)\right] = \lim_{n\to\infty} \Delta x \sum_{k=0}^{n} f(x_k) + \Delta x \sum_{k=0}^{n} g$$

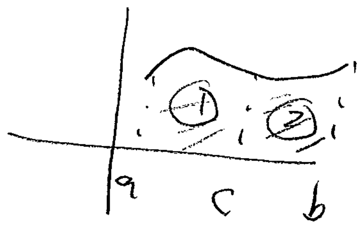
$$= \lim_{n\to\infty} \Delta x \sum_{k=0}^{n} f(x_k) + \lim_{n\to\infty} \Delta x \sum_{k=0}^{n} g(x_k) = \int_a^b f(x)\,dx + \int_a^b g(x)\,dx$$

(2) $$\int_a^b c f(x)\,dx = c \int_a^b f(x)\,dx$$

(3) $$\int_a^a f(x)\,dx = 0$$

(4) $$\int_b^a f(x)\,dx = -\int_a^b f(x)\,dx$$

(5) $$\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$$



(6) If $f(x) \le g(x)$ on $[a,b]$, then

$$\int_a^b f(x)\,dx \le \int_a^b g(x)\,dx.$$

Question: How to find the integral of a function in an easy way?!

Example 2: Using the rules of integrals, find $\int_a^b x\,dx$ if $\int_0^b x\,dx = \frac{b^2}{2}$.

Solution:

We have that $\int_0^b x\,dx = \frac{b^2}{2}$, so

$$\int_a^b x\,dx = \int_a^0 x\,dx + \int_0^b x\,dx$$

$$= -\int_0^a x\,dx + \frac{b^2}{2} = -\frac{a^2}{2} + \frac{b^2}{2} = \frac{b^2-a^2}{2}.$$

Exercise 2:

Suppose $\int_1^5 f(x)\,dx = 3$, $\int_1^3 f(x)\,dx = 1$, $\int_1^3 h(x)\,dx = 5$

Find

(1) $\int_1^5 -2f(x)\,dx$ (2) $\int_1^3 [f(x)+h(x)]\,dx$ (3) $\int_1^3 2f(x)-5h(x)\,dx$

(4) $\int_5^1 f(x)\,dx$ (5) $\int_3^5 f(x)\,dx$ (6) $\int_3^1 (h(x)-f(x))\,dx$