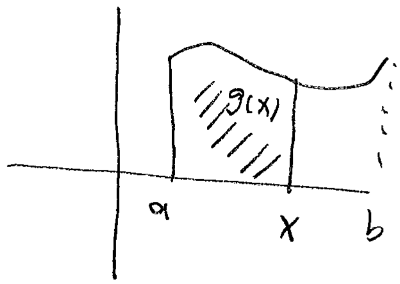
§5.4- The Fundamental Theorem of Calculus

"It establishes a connection between derivatives and integration".

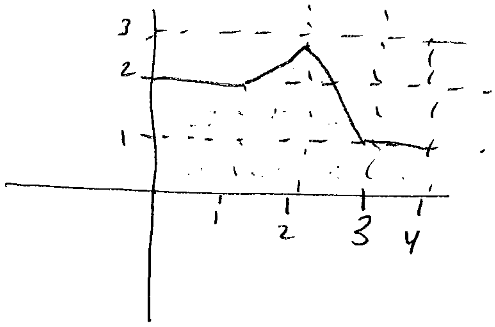
The fundamental theorem of calculus, part I

Consider the function,

$$g(x) = \int_a^x f(t)\,dt$$ 'area so far'



Example 1: Let $g(x) = \int_0^x f(t)\,dt$



- $g(0) = \int_0^0 f(t)\,dt = 0$
- $g(1) = \int_0^1 f(t)\,dt = 2$
- $g(2) = \int_0^2 f(t)\,dt = 4.5$
- $g(3) = \int_0^3 f(t)\,dt = 6.5$
- $g(4) = \int_0^4 f(t)\,dt = 8$

Question: what is the derivative of $g(t) = \int_a^x f(t)\,dt$?

Answer: FTC, part I

$$g'(t) = f(x) \qquad \left[\frac{d}{dx}\int_a^x f(t)\,dt = f(x)\right]$$

Fundamental Theorem of Calculus, part I (General form)

If $\quad g(x) = \int_{h(x)}^{g(x)} f(t)\,dt = f(g(x)) \cdot g'(x) - f(h(x))h'(x)$

$$\left[\frac{d}{dx}\int_{h(x)}^{g(x)} f(t)\,dt = f(g(x))g'(x) - f(h(x)) \cdot h'(x)\right]$$

Example 2: Find the derivative of

(1) $g(x) = \int_{1}^{x} \frac{1}{\cancel{x}^3+1}\,dt$

$$g'(\cancel{x}) = \frac{1}{x^3+1}.$$

(2) $\frac{d}{dx}\left(\int_{x}^{2} \sqrt{1+\sec t}\;dt\right) = \sqrt{1+\sec(2)} \cdot (2)' - \sqrt{1+\sec x} \cdot (x)'$

$$= -\sqrt{1+\sec x}.$$

(3) $\frac{d}{dx}\int_{0}^{\tan x} \sqrt{t+\sqrt{t}}\;dt = \sqrt{\tan x+\sqrt{\tan x}} \cdot (\tan x)' - \sqrt{0+\sqrt{0}} \cdot (0)'$

0

$$= \sqrt{\tan x+\sqrt{\tan x}} \cdot \sec^2 x$$

(4) $\frac{d}{dx}\left(\int_{0}^{x^3} e^{-t}\,dt\right) = e^{-x^3} \cdot (x^3)' - e^{-0}(0)'$

$$= e^{-x^3} \cdot 3x^2.$$

(5) $\frac{d}{dx}\left(\int_{\tan x}^{x^2} \frac{1}{\sqrt{3+t^4}}\,dt\right) = \frac{1}{\sqrt{1+x^8}}(2x) - \frac{1}{\sqrt{3+\tan^4 x}} \cdot \sec^2 x$

The Fundamental Theorem of Calculus, part II

$$\int_a^b f(x)\,dx = F(b) - F(a)$$, where $F(x)$ is the anti-derivative of $f(x)$ $[\,F'(x) = f(x)\,]$

Example 2: Evaluate each integral

(1) $$\int_{-1}^{2} (x^3 - 6x)\,dx = \Big[\overbrace{\frac{x^4}{4} - 3x^2}^{\text{anti-derivative}}\Big]_{-1}^{2} = \left(\frac{(2)^4}{4} - 3(2)^2\right) - \left(\frac{1}{4} - 3(1)\right)$$

$$= -\frac{45}{4}$$

(2) $$\int_1^{27} \sqrt[3]{x}\,dx = \Big[\,x^{\frac{4}{3}}\,\Big]_1^{27} = (27)^{\frac{4}{3}} - (1)^{\frac{4}{3}} = 81 - 1 = 80$$

(3) $$\int_0^2 x(3+x^2)\,dx = \int_0^2 3x + x^3\,dx = \Big[\frac{3}{2}x^2 + \frac{1}{4}x^4\Big]_0^2 = 10$$

(4) $$\int_0^{\frac{\pi}{4}} \sec^2 t\,dt = \big[\tan t\big]_0^{\frac{\pi}{4}} = \tan\frac{\pi}{4} - \tan 0 = 1 - 0 = 1$$

(5) $$\int_1^2 \frac{t^5 + 3t^3}{t^4}\,dt = \int_1^2 t + \frac{3}{t}\,dt = \Big[\frac{t^2}{2} + 3\ln t\Big]_1^2$$

$$= 2 + 3\ln 2 - \left(\frac{1}{2} + 3\ln 1\right) = \frac{3}{2} + 3\ln 2$$

(5) $$\int_{-2}^{2} f(x)\,dx$$, where $$f(x) = \begin{cases} 2 & , -2 \le x \le 0 \\ 4 - x^2 & , 0 < x \le 2 \end{cases}$$

$$= \int_{-2}^{0} f(x)\,dx + \int_0^2 f(x)\,dx = \int_{-2}^{0} 2\,dx + \int_0^2 4 - x^2\,dx = 2(2) + \Big[4x - \frac{x^3}{3}\Big]_0^2 = 4 + \frac{16}{3}$$

(6) $\int_0^1 \frac{2}{1+x^2}\,dx = \left[2\tan^{-1}x\right]_0^1 = 2\tan^{-1}1 - 2\tan^{-1}0 = \frac{\pi}{2}$

Exercise 4: Evaluate the integral

(1) $\int_1^{32} x^{-6/5}\,dx$ (2) $\int_0^{\pi/4} \tan x \sec x\,dx$ (3) $\int_{-1}^{1} (r+2)^2\,dr$

(4) $\int_0^{\pi} (2+\cos x)\,dx$ (5) $\int_0^{3} e^x\,dx$ (6) $\int_{-2}^{2} |x|\,dx$

Differentiation and Integration as Inverse processes

FTC1: If $g(x) = \int_a^x f(t)\,dt \rightarrow g'(x) = f(x)$

FTC2: $\int_a^b f(x)\,dx = F(b) - F(a)$, where $F'(x) = f(x)$

- $\frac{d}{dx}\left(\int_a^x f(t)\,dt\right) = f(x)$.
- $\int_a^b F'(x)\,dx = F(b) - F(a)$.

"This is considered as one of the great accomplishments of the human mind".