§5.5- Indefinite Integrals and the Substitution Method

Integral

→ Definite

$$\int_a^b f(x)\,dx$$

"Number"

→ Indefinite

$$\int_a^x f(t)\,dt \rightarrow \int f(x)\,dx = F(x) + C$$

function in x

Example 1: Evaluate the following

(i) $\int (6x^2+3x+2)\,dx = \frac{x^3}{2} + \frac{3x^2}{2} + 2x + C$

(ii) $\int \csc^2 x\,dx = -\cot x + C$

(iii) $\int 4\sin\theta + 3\cos\theta\,d\theta = -4\cos\theta + 3\sin\theta + C$.

Substitution Method:-

Idea: To replace a relatively complicated integral by a Simpler one. by adding extra variable.

$$\boxed{\int f(g(x)) \cdot g'(x)\,dx = \int f(u)\,du}$$

Example 1: Find $\int \underbrace{x}_{g'} \cos(\underbrace{x^2}_{g})\,dx$

Solution:

let $u = g(x) = x^2$

$du = 2x\,dx \longrightarrow dx = \frac{du}{2x}$.

$$\int x\cos(x^2)\,dx = \int \cancel{x}\cos(u)\cdot\frac{du}{2\cancel{x}} = \frac{1}{2}\int \cos u\,du = \frac{1}{2}\sin u + C$$

$$= \frac{1}{2}\sin(x^2) + C.$$

Exercise 1: find $\int \sqrt{3x+5}\,dx$.

Example 2: Find $\int \frac{x}{\sqrt{1-9x^2}}\,dx$

Solution:

let $u = 1-9x^2 \rightarrow du = -18x\,dx \rightarrow dx = \frac{du}{-18x}$

$$\int \frac{x}{\sqrt{1-9x^2}}\,dx = \int \frac{\cancel{x}}{\sqrt{u}}\cdot\frac{du}{-18\cancel{x}} = -\frac{1}{18}\int \frac{1}{\sqrt{u}}\,du = -\frac{1}{18}\int u^{-\frac{1}{2}}\,du$$

$$= -\frac{1}{9}u^{\frac{1}{2}} + C = -\frac{1}{9}\sqrt{u} + C = -\frac{1}{9}\sqrt{1-9x^2} + C.$$

Exercise 2: find $\int \frac{z^2}{\sqrt[3]{1+z^3}}\,dz$

Example 3: Find $\int x^3\sqrt{x^2+1}\,dx$

let $u = x^2+1 \rightarrow du = 2x\,dx \rightarrow dx = \frac{du}{2x}$

$$\int x^3\sqrt{x^2+1}\,dx = \int x^3\sqrt{u}\,\frac{du}{2x} = \frac{1}{2}\int x^2\sqrt{u}\,du$$

$$= \frac{1}{2}\int (u-1)\sqrt{u}\,du = \frac{1}{2}\int u^{\frac{3}{2}} - u^{\frac{1}{2}}\,du$$

$$= \frac{1}{2}\left[\frac{2}{5}u^{\frac{5}{2}} - \frac{2}{3}u^{\frac{3}{2}}\right] = \frac{1}{5}u^{\frac{5}{2}} - \frac{1}{3}u^{\frac{3}{2}} + c$$

$$= \frac{1}{5}(x^2+1)^{\frac{5}{2}} - \frac{1}{3}(x^2+1)^{\frac{3}{2}} + c.$$

Example 4: Find $\int_0^{\sqrt{\pi}} x\cos(x^2)\,dx$

let $u = x^2 \rightarrow du = 2x\,dx \rightarrow dx = \frac{du}{2\cancel{x}}$

$x = 0 \longrightarrow u = (0)^2 = 0$

$x = \sqrt{\pi} \rightarrow u = (\sqrt{\pi})^2 = \pi$.

$$\int_0^{\sqrt{\pi}} x\cos(x^2)\,dx = \frac{1}{2}\int_0^{\pi}\cos(u)\,du = \frac{1}{2}\Big[\sin u\Big]_0^{\pi} = \frac{1}{2}\Big[\sin\pi - \sin 0\Big]$$

$$= 0.$$

Exercise 3: Find $\int_0^{\pi}\sec^2\left(\frac{x}{4}\right)dx$.

Example 5: Find $\int_{1}^{e^2} \frac{(\ln x)^2}{x} dx$

let $u = \ln x \rightarrow du = \frac{1}{x} dx$

$x = 1 \rightarrow u = \ln 1 = 0$

$x = e^2 \rightarrow u = \ln e^2 = 2$

$$\int_{1}^{e^2} \frac{(\ln x)^2}{x} dx = \int_{0}^{2} u^2 du = \left[\frac{u^3}{3}\right]_0^2 = \frac{8}{3}.$$

Exercise 4: $\int_{\ln 2}^{\ln 3} e^x \sqrt{1+e^x} dx.$

Exercise 5: $\int_{0}^{\frac{1}{2}} \frac{\sin^{-1} x}{\sqrt{1-x^2}} dx.$

Exercise 6: $\int_{e}^{e^4} \frac{dx}{x\sqrt{\ln x}}$

Exercise 7: $\int \frac{1}{ax+b} dx \quad (a \neq 0).$