§4.1 - Exponential Functions

- Exponential Function, Graphs, and its properties
- Compound interest.
- The Euler number e.

1- The exponential function, graph, and its properties

Definition:

The function

$$f(x) = a^x \quad , \quad a>0, \ a \neq 1$$

is called an exponential function. a is called the base and x is called the exponent (power)

Recall: Rules for exponent

1. $a^x \cdot a^y = a^{x+y}$
2. $\frac{a^x}{a^y} = a^{x-y}$
3. $(a^x)^y = a^{xy}$
4. $(ab)^x = a^x b^x$
5. $a^0 = 1$
6. $a^1 = a$
7. $a^{-x} = \frac{1}{a^x}$

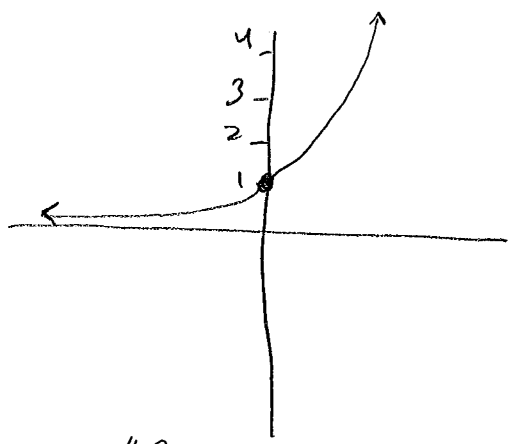
Example 1: (Graphing Exponential function with $a>1$)

Graph the function

$$f(x) = 3^x$$

Solution: using the calculator, we fill the table

x	-2	-1	0	1	2	3
y	$\frac{1}{9}$	$\frac{1}{3}$	1	3	9	27



"General Shape"

Domain $= (-\infty, \infty)$

Range $= (0, \infty)$

y-intercept $= (0, 1)$

x-intercept = none

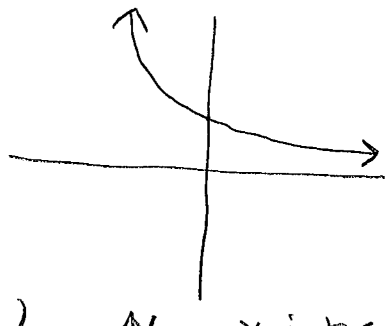
Exercise 2: Graph $f(x) = 7^x$ and observe the differences with the previous example.

Example 2: (Graphing Exponential function with $0<a<1$)

Graph $f(x) = \left(\frac{1}{3}\right)^x$

Solution:

x	-2	-1	0	1	2	3
y	9	3	1	$\frac{1}{3}$	$\frac{1}{9}$	$\frac{1}{27}$



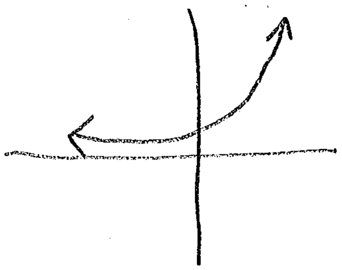
Domain $= (-\infty, \infty)$, Range $= (0, \infty)$, y-intercept $= (0, 1)$, No x-intercept.

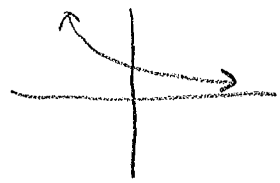
Exercise 2: Graph $f(x) = \left(\frac{1}{7}\right)^x$.

Summary:

$$y = f(x) = a^x$$

$a > 1$ $\qquad$ $0 < a < 1$





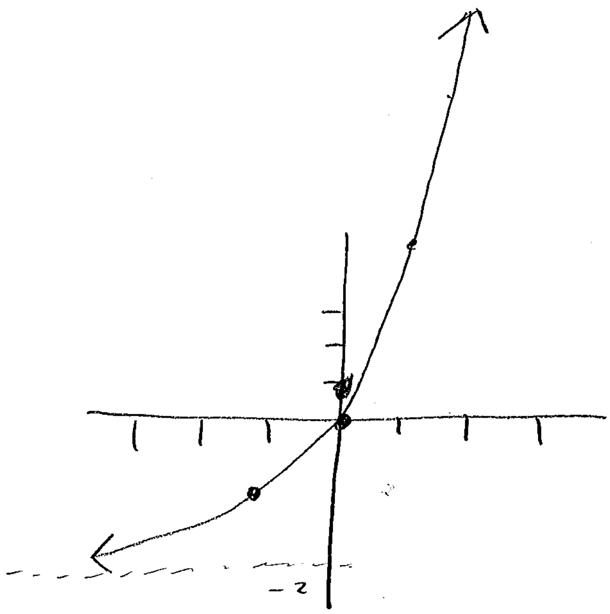
Domain $= (-\infty, \infty)$

Range $= (0, \infty)$

Example 3: Graph $f(x) = 3^{x+1} - 2$.

Solution:

x	-3	-2	-1	0	1	2	3
y	$-\frac{17}{9}$	$-\frac{5}{3}$	-1	1	7	25	79



Domain $= (-\infty, \infty)$

Range $= (-2, \infty)$

y-intercept $= (0, 1)$

2- Compound Interest

Example 4: Suppose you save 1000 BD in a saving account that pays 1% annually. Find the total money in your account every year.

Solution:

Year 0 : $A_0 = 100$ BD

Year 1 : $A_1 = \underbrace{100}_{\text{principal}} + \underbrace{100\left(\frac{1}{100}\right)}_{\text{interest}} = 101 = A_0(1+r) = 100(1+0.01)$

Year 2 : $A_2 = A_1 + A_1(0.01) = A_1(1+0.01) = 100(1+0.1)(1+0.1)$

$= 100(1+0.1)^2$

Year 3 : $A_3 = A_2 + A_2 r = A_2(1+r) = A_0(1+r)^2(1+r)$

$= A_0(1+r)^3 = 100(1+0.01)^3$

$\vdots$

Year n : $A_n = A_{n-1} + A_{n-1} r = A_{n-1}(1+r) = A_0(1+r)^n$

So in any year, we have

$$A_n = P(1+r)^n.$$

Example 5:

Suppose you saved 3000 BD at 5% for 3 years. Find the compound amount and the compound interest.

Solution:

$n = 3$, $P = 3000$, $r = 5\% = 0.05$

$$A_3 = P(1+r)^n$$
$$= 3000(1+0.05)^3$$
$$= 3472.875 \text{ BD}$$

$$I = A_3 - P = 3472.875 - 3000 = 472.875 \text{ BD}.$$

In general, if the interest are given periodically (say m times a year), the formula is

$$A_n = P\left(1+\frac{r}{m}\right)^{n \cdot m}, \quad n = \# \text{ of years}$$

Example 6: Find the compound amount and the compound interest.

(a) 500 BD for 7 years at 11% semi annually.

$P = 500$, $n = 7$, $r = 0.11$, $m = 2$.

$$A_7 = P\left(1+\frac{r}{m}\right)^{n \cdot m} = 500\left(1+\frac{0.11}{2}\right)^{(7)(2)} = 1058.04 \text{ BD}$$

(b) 4000 at 8.5% for 15 years quartely.

$P = 4000$, $r = 0.085$, $n = 15$, $m = 4$

$$A_{15} = P\left(1+\frac{r}{m}\right)^{n \cdot m} = 4000\left(1+\frac{0.085}{4}\right)^{(4)(15)} = 14124.86 \text{ BD}.$$

Exercise 3: Find the compound amount & compound interest of investing

(a) 300 BD at 7 % for 9 years compounded yearly

(b) 200 BD for 6 years at 5% monthly

(c) 1000 BD for 2 years at 9 % semi-annually

(d) 200 BD for 2 years at 1% daily (365 days in one year)

(e) (old exam question) 11 000 BD for 9 years at 3% compounded monthly

(f) (old exam question) 1020 BD for 8 years at 6% compounded monthly.

3- The Euler Function

Motivational Example:

Suppose you invest 1 BD in an account that pays 100% in every period (we have m periods). Find the compound amount for one year.

$P = 1$, $r = 100\% = 1$, $n = 1$, $m = m$.

$$A_m = 1\left(1+\frac{1}{m}\right)^m = \left(1+\frac{1}{m}\right)^m$$

Now check $m = 1, 2, 3, \ldots, 1000, 1000{,}000{,}000, \ldots$

You see your number never exceed 3!!!, as $n \to \infty$

$A_m = 2.71828\ldots$ which is the euler number e.

So $e = \lim_{m\to\infty}\left(1+\frac{1}{m}\right)^m = 2.71828\ldots$

- e is not rational, i.e., the decimal expansion of e never ends or repeat in pattern.

- $e = 1 + \frac{1}{1} + \frac{1}{2!} + \frac{1}{3!} + \frac{1}{4!} + \ldots$

Example 7: Find the value of

(a) $e^{2.5}$ (b) e^{-1} (c) $e^{-\frac{1}{3}}$.

Example 7: Graph the function $y = -e^{-x+3}$

x	-2	-1	0	1	2	3
y	-148.4	-54.5				-1