§ 4.3 - Properties of Logarithms

1. $\log_a (m \cdot n) = \log_a m + \log_a n$

2. $\log_a \left(\frac{m}{n} \right) = \log_a m - \log_a n$

3. $\log_a m^r = r \log_a m$

4. $\log_a 1 = 0$

5. $\log_a a = 1$

6. (change of base) $\log_a m = \dfrac{\log_b m}{\log_b a}$.

Proof: (Not Required).

(1) Recall the fundamental Equations

$$a^{\log_a x} = x \quad \text{— (1)}$$

$$\log_a a^x = x \quad \text{— (2)}$$

Now compute $a^{\log_a m + \log_a n} = a^{\log_a m} \cdot a^{\log_a n}$

$= m \cdot n$ — by (1)

So $a^{\log_a m + \log_a n} = m \cdot n$, write it in the logarithmic form, we get

$$\boxed{\log_a m + \log_a n = \log_a m \cdot n}$$

(2) (Exercise)

(3) (Exercise) Compute $a^{r\log_a m}$.

(4) $\log_a 1 = \cancel{x} \longrightarrow 1 = a^x \rightarrow a^0 = a^x$, so $x=0$

$$\boxed{\log_a 1 = 0}$$

(5) $\log_a a = x \longrightarrow a^1 = a^x \rightarrow x = 1$, hence

$$\boxed{\log_a a = 1}$$

(6) Compute $b^{(\log_a m)(\log_b a)} = \left(b^{\log_b a}\right)^{(\log_a m)} = \left(a^{\log_a m}\right) = m$

So in exponential form

$$\boxed{(\log_a m)(\log_b a) = \log_b m}$$

Example 1: Let $\log 2 = a$, $\log 3 = b$, $\log 5 = c$. Find in term of $a, b,$ and c

1. $\log 6 = \log(2\cdot 3) \underset{\text{(Rule 1)}}{=} \log 2 + \log 3 = a + b$

2. $\log 15 = \log(3\cdot 5) = \log(3) + \log(5) = b + c$

3. $\log 60 = \log(2^2\cdot 3\cdot 5) = \log 2^2 + \log 3 + \log 5$

$= 2\log 2 + \log 3 + \log 5$

$= 2a + b + c$

4. $\log_2 3 \underset{\text{(Rule 7)}}{=} \frac{\log 3}{\log_{10} 2} = \frac{b}{a}$

5. $\log 1000 = \log(2^3\cdot 5^3) = \log 2^3 + \log 5^3 = 3a + 3b$.

Exercise 1: In the previous example, Find

(1) $\log_5 3$ (2) $\log 10$ (3) $\log 0.00002$.

(4) $\log \frac{25}{6}$.

Exercise 2: If $\log_a 5 = 0.83$, $\log_a 3 = 0.56$, find

(a) $\log_a 15$ (b) $\log_a 25$ (c) $\log_a(\sqrt{3})$.

Example 2: (expanded) write the following expressions as sum or difference of logarithms

(a) $\ln \dfrac{x}{wz^2} = \ln x - \ln(wz^2)$

$= \ln x - (\ln w + \ln z^2) = \ln x - \ln w - 2\ln z.$

(b) $\ln\left(\dfrac{x+1}{x+5}\right)^4 = 4\ln\left(\dfrac{x+1}{x+5}\right) = 4\ln(x+1) - 4\ln(x+5).$

(c) $\ln \dfrac{\sqrt{x}}{(x^2)(x+3)^4} = \ln\sqrt{x} - \ln x^2 - \ln(x+3)^3$

$= \ln x^{\frac{1}{2}} - 2\ln x - 3\ln(x+3)$

$= \frac{1}{2}\ln x - 2\ln x - 3\ln(x+3)$

$= -\frac{3}{2}\ln x - 3\ln(x+3).$

Exercise 3: write each expression as sum of difference of logarithms

(a) $\log_3\left(\dfrac{5\cdot 7}{4}\right)$ (b) $\log_2\left(\dfrac{x^5}{y^2}\right)$ (c) $\log\left(\dfrac{x^2 z}{wy^2}\right)$.

Example 3: (single logarithm) write each of the following as a single logarithm.

(1) $\log 6 + \log 4 = \log(6\cdot 4) = \log(24)$

(2) $2\log x - \frac{1}{2}\log(x-3) = \log x^2 - \log(x-3)^{\frac{1}{2}} = \log \dfrac{x^2}{(x-3)^{\frac{1}{2}}}$.

3) $2 + 10\log 3 = 2\log 10 + 10\log 3 = \log 10^2 + \log 3^{10} = \log(10^2\cdot 3^{10})$

(4) $\log_3\sqrt{3} - \log_2\sqrt[3]{2} + \log_7\sqrt[5]{7} = 1 = \log_3 3^{\frac{1}{2}} - \log_2 2^{\frac{1}{3}} + \log_7 7^{\frac{1}{5}} = \frac{1}{2}\log_3 3 - \frac{1}{3}\log_2 2 + \frac{1}{5}\log_7 7 = \frac{1}{2} - \frac{1}{3} + \frac{1}{5}$.

Exercise 4: Write each of the following as a single logarithm.

(a) $2\log_5 3 + 3\log_5 2 = \log_5 3^2 + \log_5 2^3 = \log_5 3^2 \cdot 2^3$.

(b) $3\log_a x - \log_a (x+1) = \log_a x^3 - \log_a (x+1) = \log_a \frac{x^3}{x+1}$.

(c) $\log_4 25 + \log_4 3 - \log_4 5 = \log_4 \frac{25\cdot 3}{5} = \log_4 15$.

(d) $\log_5 8 - \log_5 x$

(e) $\log_{10} 27 - \log_{10} 3$ (g) 1

(f) $\log_3 (x^2+5) - \log_3 (x^2+1)$.

Recall: The Fundamental Equations

$$a^{\log_a x} = x \qquad \& \qquad \log_a a^x = x$$

Example 8: Find the value of the following:

(1) $\log_5 5^{212} = 212 \log_5 5 = 212$

(2) $\ln e^{0.1} = 0.1 \ln e = 0.1$

(3) $\log \frac{1}{10} + \ln e^3 = \log 10^{-3} + 3 \ln e^3 = -3 \underbrace{\log 10}_{=1} + 3 \underbrace{\ln e}_{=1} = -3+3=0$

(4) $e^{\ln 5} = 5$ (by the fundamental Equation).