Chapter 6: Matrix Algebra

Algebra

Numbers
- Set of Numbers
- operations $(+, \cdot, -, \div)$

Functions
- set of functions
- operations $(+, -, \cdot, \div, \circ)$
  composite

Matrices (plural)
- Set of matrices
- operations $(+, -, \cdot, \div)$, ...

We want to learn:

1- What a matrix is?

2- How to add two matrices?

3- How to multiply two matrices?

4- How to find the multiplicative inverse?

5- What is the determinant of a matrix and why it is useful?

6- How to solve system of linear equations using matrices?

§ 6.1 - Matrices

• A matrix (singular) (plural): matrices "may tri sees") is just a rectangular array of numbers (entries). They have rows and columns.

Example:

$$A = \begin{array}{c c} & \begin{array}{cc} \text{column 1} & \text{column 2} \end{array} \\ \begin{array}{c} \text{row 1} \\ \text{row 2} \end{array} & \begin{pmatrix} 2 & 0 \\ 3 & 2 \end{pmatrix} \end{array}, \quad B = \begin{pmatrix} 5 & 6 & 1 \\ 7 & 1 & 2 \end{pmatrix}, \quad C = \begin{array}{cc} \begin{array}{ccc} 1 & 2 & 3 \end{array} & \\ \begin{pmatrix} 1 & 2 & 3 \end{pmatrix} & \leftarrow \text{row 1} \end{array}$$

2 X 3   1 X3.

Size = 2×3

$\uparrow$ # of rows $\uparrow$ # of columns

Example:

$$\begin{pmatrix} B_{11} & B_{12} & B_{13} \\ B_{21} & B_{22} & B_{23} \end{pmatrix}, \quad B_{ij}$$

i: row, j: column

Definition:

An $m \times n$ - matrix is a rectangular array consists of m rows and n columns

$$A = \begin{pmatrix} A_{11} & A_{12} & A_{13} & \cdots & A_{1n} \\ A_{21} & A_{21} & A_{23} & \cdots & A_{2n} \\ \vdots & & & & \\ A_{m1} & A_{m2} & A_{m3} & \cdots & A_{mn} \end{pmatrix},$$

where A_{ij} is the entry in the row i and column j, usually we write $A = (A_{ij})_{m \times n}$

Example 1: let

$$A = (A_{ij}) = \begin{pmatrix} 3 & -2 & 7 & 3 \\ 2 & 1 & -1 & -5 \\ 4 & 3 & 2 & 1 \\ 0 & 8 & 0 & 2 \end{pmatrix}$$

(1) what is the size of A?

(2) Find A_{21}, A_{42}, A_{32}, A_{23}, A_{34}, A_{44}, A_{55}.

(3) What are the entries of the second row?

Definition: (Transpose of a matrix)

If A is a matrix, the transpose of A is a new matrix A^T formed by interchanging the rows and the columns of A, i.e.,

$$A^T = (A_{ji}).$$

Example 2: Find A^T and $(A^T)^T$

$$A = \begin{pmatrix} 6 & -3 \\ 2 & 4 \end{pmatrix}, \quad B = \begin{pmatrix} 2 & 1 & 3 \\ 7 & 1 & 6 \end{pmatrix}, \quad C = (3\ 1\ 2\ 5).$$

Solution:

- $A^T = \begin{pmatrix} 6 & 2 \\ -3 & 4 \end{pmatrix}$ and $(A^T)^T = \begin{pmatrix} 6 & -3 \\ 2 & 4 \end{pmatrix} (= A)$

- $B^T = \begin{pmatrix} 2 & 7 \\ 1 & 1 \\ 3 & 6 \end{pmatrix}$ and $(B^T)^T = \begin{pmatrix} 2 & 1 & 3 \\ 7 & 1 & 6 \end{pmatrix}$

- $C^T = \begin{pmatrix} 3 \\ 1 \\ 2 \\ 5 \end{pmatrix}$ and $(C^T)^T = (3\ 1\ 2\ 5)$

Note:

1- $(A^T)^T = A$.

2- A matrix is called <u>symmetric</u> if $A^T = A$.

Question: When two matrices are equal?

Definition: Two matries A & B are <u>equal</u> if they have the same size and same entries at the same position, i.e.

$$A_{ij} = B_{ij}.$$

Example 3: Solve the matrix equation

$$\begin{pmatrix} 4 & 2 & 1 \\ x & 2y & 3z \\ 0 & 1 & 2 \end{pmatrix} = \begin{pmatrix} 4 & 2 & 1 \\ -3 & -8 & 0 \\ 0 & 1 & 2 \end{pmatrix}$$

Solution:

$x = -3$, ~~and~~ $2y = -8$, and $3z = 0$

$x = -3$, $y = -4$, and $z = 0$

Solution set $= \{(-3, -4, 0)\}$.

Special Matrices

- <u>Zero matrix</u> $O_{m\times n} = (0)_{m\times n}$, "zero everywhere"

$$\begin{pmatrix} 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \ldots$$

- <u>Square matrix</u> if $m=n$ (have the same ~~#~~ rows & columns)

$$\begin{pmatrix} 3 & 2 \\ 1 & 7 \end{pmatrix}, \quad \begin{pmatrix} 5 & 1 & 2 \\ 3 & 1 & 5 \\ 5 & 5 & 1 \end{pmatrix}, \quad (5), \quad \ldots$$

main Diagonal   Main Diagonal

- <u>Diagonal matrix</u> if ~~it~~ is a square matrix with all the entries off the main diagonal is zero

$$\begin{pmatrix} 3 & 0 \\ 0 & 5 \end{pmatrix}, \quad \begin{pmatrix} 7 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix}, \quad \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 5 \end{pmatrix}, \quad \ldots$$

- <u>Upper triangular matrix</u> is having zero below the main diagonal

$$\begin{pmatrix} 5 & 7 & 6 \\ 0 & 2 & 5 \\ 0 & 0 & 1 \end{pmatrix}, \quad \begin{pmatrix} 3 & 1 \\ 0 & 2 \end{pmatrix}, \quad \ldots$$

U

- <u>Lower triangular matrix</u> is having zero above the main diagonal (entries are lower the main diagonal)

$$\begin{pmatrix} 5 & 0 & 0 \\ 6 & 2 & 0 \\ 3 & 1 & 7 \end{pmatrix}, \quad \begin{pmatrix} 3 & 0 \\ 1 & 2 \end{pmatrix}, \quad \ldots$$

- <u>row vector</u> is a matrix with only <u>one</u> row. $(3\ 2\ 1\ 7)$
- <u>column vector</u> is a matrix with only <u>one</u> column $\begin{pmatrix} 3 \\ 5 \\ 7 \end{pmatrix}$
- <u>Identity matrix</u> is a matrix that has 1 in the main diagonal and zero elsewhere

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad (1), \quad \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \ldots$$