§ 6.4 - Solving Systems by Row operations

Goal: To solve system of linear equations by using elementary row operations on matrices.

1- Augmented Matrix

Example 1: consider the system of linear equations

$$2x - y = 5$$
$$x + 3y = -1$$

The augmented matrix of this system is

$$\begin{array}{cc} x & y \\ \downarrow & \downarrow \end{array}$$
$$\left(\begin{array}{cc|c} 2 & -1 & 5 \\ 1 & 3 & -1 \end{array}\right)$$

2- Elementary row operations

1- Interchanging any two rows ($R_i \leftrightarrow R_j$)

2- Multiplying (Dividing) a row by non-zero number ($R_i \xrightarrow[\text{"becomes"}]{} cR_i$)

3- Add a multiply of a row to another row ($R_i \underset{\text{"becomes"}}{\rightarrow} R_i + cR_j$)

Example 2: $\begin{pmatrix} 1 & 0 & 2 \\ 2 & 5 & 1 \\ 3 & 0 & -2 \end{pmatrix}$, $R_3 \rightarrow R_3 + 2R_1$

$$\begin{pmatrix} 1 & 0 & 2 \\ 2 & 5 & 1 \\ 3+2(1) & 0+2(0) & -2+2(2) \end{pmatrix} = \begin{pmatrix} 1 & 0 & 2 \\ 2 & 5 & 1 \\ 5 & 0 & 2 \end{pmatrix}$$

Goal of elementary row operations

We want to reach a <u>reduced</u> <u>matrix</u>, which is a matri

1- All zero-rows are at the bottom.

2- Each non-zero row has a leading 1's (<u>Pivot</u>) and all other entries in the pivot column is zero.

3. The pivots start from left to right.

<u>Example 3:</u> There are in the reduced forms

$$\begin{pmatrix} 1 & 3 & 0 & 5 & 1 \\ 0 & 0 & 1 & 2 & 6 \end{pmatrix}, \quad \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \begin{pmatrix} 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad \underbrace{\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}}_{\text{Not!}}$$

<u>Example 3:</u> Reduce the matrix

$$\begin{pmatrix} 0 & -3 & 0 & 2 \\ 1 & 5 & 0 & 2 \end{pmatrix}$$

<u>Solution:</u>

$$\begin{pmatrix} 0 & -3 & 0 & 2 \\ 1 & 5 & 0 & 2 \end{pmatrix}, \quad R_1 \Leftrightarrow R_2$$

$$\begin{pmatrix} 1 & 5 & 0 & 2 \\ 0 & -3 & 0 & 2 \end{pmatrix}, \quad R_2 \to -\frac{1}{3} R_2$$

$$\begin{pmatrix} 1 & 5 & 0 & 2 \\ 0 & 1 & 0 & 7\frac{2}{3} \end{pmatrix}, \quad R_1 \to R_1 - \frac{5}{1} R_2$$

$$\begin{pmatrix} 1-5(0) & 5-5(1) & 0-5(0) & 2-5\left(-\frac{2}{3}\right) \\ 0 & 1 & 0 & -\frac{2}{3} \end{pmatrix}$$

$$\begin{pmatrix} 1 & 0 & 0 & \frac{16}{3} \\ 0 & 1 & 0 & -\frac{2}{3} \end{pmatrix}.$$

which is a <u>reduced matrix</u>.

<u>3 - Solving System of linear equation by row operations</u>

<u>Example 4:</u> Solve

$$2x - 7y = -1$$
$$x + 3y = 6$$

<u>Solution:</u>

$$\left(\begin{array}{cc|c} 2 & -7 & -1 \\ 1 & 3 & 6 \end{array}\right), \quad R_1 \leftrightarrow R_2$$

$$\left(\begin{array}{cc|c} 1 & 3 & 6 \\ 2 & -7 & -1 \end{array}\right), \quad R_2 \to R_2 - \frac{2}{1} R_1$$

make it zero

$$\left(\begin{array}{cc|c} 1 & 3 & 6 \\ 2-2(1) & -7-2(3) & -1-2(6) \end{array}\right)$$

$$\left(\begin{array}{cc|c} 1 & 3 & 6 \\ 0 & -13 & -13 \end{array}\right), \quad R_2 \to \frac{+1}{-13} R_2$$

make this zero

$$\left(\begin{array}{cc|c} 1 & 3 & 6 \\ 0 & 1 & 1 \end{array}\right), \quad R_1 \to R_1 - \frac{3}{1} R_2$$

$$\left(\begin{array}{cc|c} 1-3(0) & 3-3(1) & 6-3(1) \\ 0 & 1 & 1 \end{array}\right)$$

$$\left(\begin{array}{cc|c} 1 & 0 & 3 \\ 0 & 1 & 1 \end{array}\right)$$

$$\begin{array}{l} x = 3 \\ y = 1 \end{array} \Rightarrow \text{Solution Set} = \{(3,1)\}.$$

<u>Example 5:</u>

$$\begin{aligned} x + 4y &= 9 \\ 3x - y &= 6 \\ 2x - 2y &= 4 \end{aligned}$$

<u>Solution:</u>

make it zero →

make it zero →

$$\left(\begin{array}{cc|c} 1 & 4 & 9 \\ 3 & -1 & 6 \\ 2 & -2 & 4 \end{array}\right), \quad \begin{array}{l} R_2 \to R_2 - 3R_1 \\ R_3 \to R_3 - 2R_1 \end{array}$$

$$\left(\begin{array}{cc|c} 1 & 4 & 9 \\ 3-3(1) & -1-3(4) & 6-3(9) \\ 2-2(1) & -2-2(4) & 4-2(9) \end{array}\right)$$

$$\left(\begin{array}{cc|c} 1 & 4 & 9 \\ 0 & -13 & -21 \\ 0 & -10 & -14 \end{array}\right), \quad R_2 \to \frac{-1}{13} R_2$$

$$\left(\begin{array}{cc|c} 1 & 4 & 9 \\ 0 & 1 & \frac{21}{13} \\ 0 & -10 & -14 \end{array}\right), \quad R_3 \to R_3 - (-10) R_2$$

make it →

$$\left(\begin{array}{cc|c} 1 & 4 & 9 \\ 0 & 1 & \frac{21}{13} \\ \hline 0 & 0 & 76 \end{array}\right) \longrightarrow 0 = 76 \quad \text{false!}$$

No Solution.

Example 6: Solve

$$x + y - z = 7$$
$$4x + 6y - 4z = 8$$
$$x - y - 5z = 23$$

Solution:

make it zero

make it zero

$$\left(\begin{array}{ccc|c} 1 & 1 & -1 & 7 \\ 4 & 6 & -4 & 8 \\ 1 & -1 & -5 & 23 \end{array}\right), \; R_2 \to R_2 - 4R_1, \; R_3 \to R_3 - R_1$$

$$\left(\begin{array}{ccc|c} 1 & 1 & -1 & 7 \\ 4-4(1) & 6-4(1) & -4-4(-1) & 8-4(7) \\ 1-1(1) & -1-(1) & -5-(-1) & 23-7 \end{array}\right)$$

$$\left(\begin{array}{ccc|c} 1 & 1 & -1 & 7 \\ 0 & 2 & 0 & -20 \\ 0 & -2 & -4 & 17 \end{array}\right), \; R_2 \to \tfrac{1}{2} R_2$$

make this zero

$$\left(\begin{array}{ccc|c} 1 & 1 & -1 & 7 \\ 0 & 1 & 0 & -10 \\ 0 & -2 & -4 & 17 \end{array}\right), \; R_3 \to R_3 + 2R_1$$

$$\left(\begin{array}{ccc|c} 1 & 1 & -1 & 7 \\ 0 & 1 & 0 & -10 \\ 0 & 0 & -4 & -3 \end{array}\right), \; R_3 \to -\tfrac{1}{4} R_3$$

make this zero

$$\left(\begin{array}{ccc|c} 1 & 1 & -1 & 7 \\ 0 & 1 & 0 & -10 \\ 0 & 0 & 1 & \frac{3}{4} \end{array}\right), \quad R_1 \to R_1 + R_3$$

make it zero

$$\left(\begin{array}{ccc|c} 1 & 1 & 0 & \frac{31}{4} \\ 0 & 1 & 0 & -10 \\ 0 & 0 & 1 & \frac{3}{4} \end{array}\right), \quad R_1 \to R_1 - R_2$$

$$\left(\begin{array}{ccc|c} 1 & 0 & 0 & \frac{71}{4} \\ 0 & 1 & 0 & -10 \\ 0 & 0 & 1 & \frac{3}{4} \end{array}\right)$$

Example 7: (Infinitely many solutions)

$$3x + 2y - z = 5$$

$$6x - y = 3$$

Solution:

$$\left(\begin{array}{ccc|c} 3 & 2 & -1 & 5 \\ 6 & -1 & 0 & 3 \end{array}\right), \quad R_1 \to \frac{1}{3} R_1$$

$$\left(\begin{array}{ccc|c} 1 & \frac{2}{3} & -\frac{1}{3} & \frac{5}{3} \\ 6 & -1 & 0 & 3 \end{array}\right), \quad R_2 \to R_2 - 6R_1$$

$$\left(\begin{array}{ccc|c} 1 & \frac{2}{3} & -\frac{1}{3} & \frac{5}{3} \\ 0 & -5 & 2 & -7 \end{array}\right), \quad R_2 \to \frac{1}{-5} R_2$$

$$\left(\begin{array}{ccc|c} 1 & \frac{2}{3} & -\frac{1}{3} & \frac{5}{3} \\ 0 & 1 & -\frac{2}{5} & \frac{7}{5} \end{array}\right), \quad R_1 \to R_1 - \frac{2}{3} R_2$$

$$\left(\begin{array}{ccc|c} 1 & 0 & \sim & \sim \\ 0 & 1 & -\frac{2}{5} & \frac{7}{5} \end{array}\right)$$

$$x + \sim z = \sim$$

$$y - \frac{2}{5} z = \frac{7}{5}$$

So $z = t \rightarrow \boxed{\begin{array}{l} x = -\sim t + \sim \\ y = \frac{7}{5} + \frac{2}{5} t \end{array}}$

parameter

Example 8: Solve

$$x + 3y = 2$$

$$2x + 7y = 4$$

$$3x + 15y + 3z = 15$$

Solution:

make these zero

$$\left(\begin{array}{ccc|c} 1 & 3 & 0 & 2 \\ 2 & 7 & 0 & 4 \\ 3 & 15 & 3 & 15 \end{array}\right), \quad \begin{array}{l} R_2 \rightarrow R_2 - 2R_1 \\ R_3 \rightarrow R_3 - 3R_1 \end{array}$$

$$\left(\begin{array}{ccc|c} 1 & 3 & 0 & 2 \\ 0 & 1 & 0 & 0 \\ 0 & 6 & 3 & 9 \end{array}\right), \quad R_3 \rightarrow R_3 - 6R_2$$

make this zero

$$\left(\begin{array}{ccc|c} 1 & 3 & 0 & 2 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 3 & 9 \end{array}\right), \quad R_3 \rightarrow \frac{1}{3} R_3$$

$$\begin{array}{c} \text{make this zero} \end{array}$$

$$\left(\begin{array}{ccc|c} 1 & 3 & 0 & 2 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 3 \end{array}\right), \quad R_1 \to R_1 - 3R_2$$

$$\left(\begin{array}{ccc|c} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 3 \end{array}\right)$$

$$\boxed{x = 2, \quad y = 0, \quad z = 3}$$

<u>Exercise</u> 2 (old Exam Question)

Solve the system

$$2x - 3y = 9$$
$$4x + y = 4$$