§ 6.6 - Inverse of a matrix

1 - Definition of the inverse.

2 - How to find the inverse

3 - Solving linear system using the inverse.

1 - Definition of the inverse

Recall:

- If a is a real number, then the additive inverse of a is $-a$ such that

$$a + (-a) = 0 \quad \text{and} \quad (-a) + a = 0$$

- If a is a non-zero real number, then the multiplicative inverse of a is $\frac{1}{a}$ such that

$$a \cdot \left(\frac{1}{a}\right) = 1 \quad \text{and} \quad \left(\frac{1}{a}\right) \cdot a = 1$$

- If f is a function that passes the horizontal line test, then the inverse of f is f^{-1} such that

$$(f \circ f^{-1})(x) = x \quad \text{and} \quad (f^{-1} \circ f)(x) = x.$$

Definition:

If A is an $n\times n$-matrix, then the inverse matrix (if it exists) of A is another $n \times n$-matrix A^{-1} such that

$$A \cdot A^{-1} = I_n \quad \text{and} \quad A^{-1} \cdot A = I_n$$

matrix multiplication §6.3

2- How to find the inverse of a matrix

write $\left(\underbrace{A}_{n\times n} \;\middle|\; \underbrace{I_n}_{n\times n}\right)$ ☆ and reduce it to get

$$\left(I_n \;\middle|\; A^{-1} \right)$$

this is the inverse

Note: If we can't reduce ☆, then the matrix has no inverse!

Example 1: Find A^{-1} of $A = \begin{pmatrix} 3 & 1 \\ 4 & 1 \end{pmatrix}$

Solution:

make this 1

$$\left(\begin{array}{cc|cc} 3 & 1 & 1 & 0 \\ 4 & 1 & 0 & 1 \end{array}\right), \quad R_1 \to \tfrac{1}{3} R_1$$

make this zero

$$\left(\begin{array}{cc|cc} 1 & \frac{1}{3} & \frac{1}{3} & 0 \\ 4 & 1 & 0 & 1 \end{array}\right), \quad R_2 \to R_2 - 4R_1$$

$$\left(\begin{array}{cc|cc} 1 & \frac{1}{3} & \frac{1}{3} & 0 \\ 0 & -\frac{1}{3} & -\frac{4}{3} & 1 \end{array}\right), \quad R_2 \to -3R_1$$

$$\left(\begin{array}{cc|cc} 1 & \frac{1}{3} & \frac{1}{3} & 0 \\ 0 & 1 & 4 & -3 \end{array}\right), \quad R_p \to R_1 - \tfrac{1}{3} R_2$$

$$\left(\begin{array}{cc|cc} 1 & 0 & \cancel{0}\,1 & 1 \\ 0 & 1 & 4 & -3 \end{array}\right) \rightsquigarrow A^{-1} = \begin{pmatrix} -1 & 1 \\ 4 & -3 \end{pmatrix}.$$

Example 2: Find the inverse of $A = \begin{pmatrix} 1 & 1 \\ 3 & 3 \end{pmatrix}$

$$\left(\begin{array}{cc|cc} 1 & 1 & 1 & 0 \\ 3 & 3 & 0 & 1 \end{array}\right), \quad R_2 \to R_2 - 3R_1$$

$$\left(\begin{array}{cc|cc} 1 & 1 & 1 & 0 \\ 0 & 0 & -3 & 1 \end{array}\right) \quad \text{so } A \text{ has } \underline{\text{No}} \text{ inverse.}$$

No inverse

Exercise 1: Find the inverse of $A = \begin{pmatrix} 1 & 0 \\ a & 1 \end{pmatrix}$, $a \in (-\infty, \infty)$

Example 3: Find the inverse of $A = \begin{pmatrix} 2 & 1 & 0 \\ 4 & -1 & 5 \\ 1 & -1 & 2 \end{pmatrix}$

Solution:

$$\left(\begin{array}{ccc|ccc} 2 & 1 & 0 & 1 & 0 & 0 \\ 4 & -1 & 5 & 0 & 1 & 0 \\ 1 & -1 & 2 & 0 & 0 & 1 \end{array}\right), \quad R_1 \leftrightarrow R_3$$

$$\left(\begin{array}{ccc|ccc} 1 & -1 & 2 & 0 & 0 & 1 \\ 4 & -1 & 5 & 0 & 1 & 0 \\ 2 & 1 & 0 & 1 & 0 & 0 \end{array}\right), \quad \begin{array}{l} R_2 \to R_2 - 4R_1 \\ R_3 \to R_3 - 2R_1 \end{array}$$

$$\left(\begin{array}{ccc|ccc} 1 & -1 & 2 & 0 & 0 & 1 \\ 0 & 3 & -3 & 0 & 1 & -4 \\ 0 & 1 & -4 & 1 & 0 & -2 \end{array}\right), \quad R_2 \leftrightarrow R_3$$

$$\left(\begin{array}{ccc|ccc} 1 & -1 & 2 & 0 & 0 & 1 \\ 0 & 1 & -4 & 1 & 0 & -2 \\ 0 & 3 & -3 & 0 & 1 & -4 \end{array}\right), \quad R_3 \to R_3 - 3R_2$$

$$\left(\begin{array}{ccc|ccc} 1 & -1 & 2 & 0 & 0 & 1 \\ 0 & 1 & -4 & 1 & 0 & -2 \\ 0 & 0 & 9 & -2 & 1 & 2 \end{array}\right), \quad R_3 \to \frac{1}{9} R_3$$

$$\left(\begin{array}{ccc|ccc} 1 & -1 & 2 & 0 & 0 & 1 \\ 0 & 1 & -4 & 1 & 0 & -2 \\ 0 & 0 & 1 & \frac{-2}{9} & \frac{1}{9} & \frac{2}{9} \end{array}\right), \quad \begin{array}{l} R_1 \to R_1 - 2R_3 \\ R_2 \to R_2 + 4R_3 \end{array}$$

$$\left(\begin{array}{ccc|ccc} 1 & -1 & 0 & \frac{4}{9} & \frac{-2}{9} & \frac{5}{9} \\ 0 & 1 & 0 & \frac{-7}{9} & \frac{4}{9} & \frac{-10}{9} \\ 0 & 0 & 1 & \frac{-2}{9} & \frac{1}{9} & \frac{2}{9} \end{array}\right), \quad R_1 \to R_1 + R_2$$

$$\left(\begin{array}{ccc|ccc} 1 & 0 & 0 & -\frac{3}{9} & \frac{2}{9} & -\frac{5}{9} \\ 0 & 1 & 0 & -\frac{7}{9} & \frac{4}{9} & \frac{-10}{9} \\ 0 & 0 & 1 & \frac{-2}{9} & \frac{1}{9} & \frac{2}{9} \end{array}\right)$$

So $$A^{-1} = \begin{pmatrix} -\frac{3}{9} & \frac{2}{9} & -\frac{5}{9} \\ -\frac{7}{9} & \frac{4}{9} & \frac{-10}{9} \\ \frac{-2}{9} & \frac{1}{9} & \frac{2}{9} \end{pmatrix} = \frac{1}{9} \begin{pmatrix} -3 & 2 & -5 \\ -7 & 4 & -10 \\ -2 & 1 & 2 \end{pmatrix}$$

3- Solving System of linear equations using matrices

To solve $A\underline{X} = b \rightsquigarrow \underline{X} = A^{-1}b$.

Example 4: Solve

$$3X + y = 2$$
$$4X + y = 3$$

Solution:

This system can be written in matrix form as

$$\begin{pmatrix} 3 & 1 \\ 4 & 1 \end{pmatrix}\begin{pmatrix} X \\ y \end{pmatrix} = \begin{pmatrix} 2 \\ 3 \end{pmatrix}$$

$$A\begin{pmatrix} X \\ y \end{pmatrix} = \begin{pmatrix} 2 \\ 3 \end{pmatrix} \rightarrow A^{-1}A\begin{pmatrix} X \\ y \end{pmatrix} = A^{-1}\begin{pmatrix} 2 \\ 3 \end{pmatrix}$$

$$\begin{pmatrix} X \\ y \end{pmatrix} = A^{-1}\begin{pmatrix} 2 \\ 3 \end{pmatrix} = \underbrace{\begin{pmatrix} -1 & 1 \\ 4 & -3 \end{pmatrix}}_{\text{Example 1}}\begin{pmatrix} 2 \\ 3 \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}.$$

Exercise 2: (old Exam Question)

The supply and demand equation of a certain product are

Supply : $2q + p = 50$

Demand : $3q - 5p = 10$

Use the matrix inverse method to find the market equilibrium point.

Example 6: Solve

$$
\begin{aligned}
2x + y &= 5 \\
4x - y + 5z &= -1 \\
x - y + 2z &= 2
\end{aligned}
$$

Solution:

$$\underbrace{\begin{pmatrix} 2 & 1 & 0 \\ 4 & -1 & 5 \\ 1 & -1 & 2 \end{pmatrix}}_{A} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 5 \\ -1 \\ 2 \end{pmatrix} \longrightarrow A \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 5 \\ -1 \\ 2 \end{pmatrix}$$

$$A^{-1} A \begin{pmatrix} x \\ y \\ z \end{pmatrix} = A^{-1} \begin{pmatrix} 5 \\ -1 \\ 2 \end{pmatrix}$$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = A^{-1} b = \frac{1}{9} \begin{pmatrix} -3 & 2 & -5 \\ -7 & 4 & -10 \\ -2 & 1 & 2 \end{pmatrix} \begin{pmatrix} 5 \\ -1 \\ 2 \end{pmatrix} = \frac{1}{9} \begin{pmatrix} -27 \\ -59 \\ 7 \end{pmatrix} = \begin{pmatrix} -3 \\ -\frac{59}{9} \\ \frac{7}{9} \end{pmatrix}.$$