Section 3.1
Lines

Dr. Abdulla Eid

College of Science

MATHS 103: Mathematics for Business I

Introduction

Recall: If we have a linear function $y = ax + b$,then by plotting two points on the graph of this function and connecting them, we get the graph of the function which is a line. (See Section 2.5)

Goal: Given two points on the line (i.e., we are given (x_1, y_1) and (x_2, y_2)). Find the equation of the line.

Note: We will need only one point (x_1, y_1) and the slope m of the line.
Topics:

1. Slope.
2. Equation of a line (multiple forms).
3. Parallel and Perpendicular lines.

1 - The slope of a line

1. The **slope** of a line is a number that measures how sloppy the line is (how hard to climb the stairs!).

1. Consider the two lines L_1 and L_2 (both of positive slope),but you can see that L_1 has slope greater than L_2.

2. Slope has a clear relation with the angle between the line and the x–axis. if the slope rises, then θ rises too!.

Finding the slope of a line

1. From the equation of the line: Solve the equation for y, i.e., let y be alone. Then, you get
$$y = mx + b$$
and the slope is m.
2. From the graph of the line: Choose any two points (x_1, y_1) and (x_2, y_2) on the line. Then,
$$m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{\text{Vertical change}}{\text{Horizontal change}}$$

Special Case: The vertical line **has no** slope. Why?

Example

Find the slope of the line that passes through
(1) $(3, -1)$ and $(6, 9)$.
(2) $(-6, 7)$ and $(0, 1)$.

Solution:
(1) $(\underbrace{3}_{x_1}, \underbrace{-1}_{y_1})$ and $(\underbrace{6}_{x_2}, \underbrace{9}_{y_2})$.

$$m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{9 - (-1)}{6 - 3} = \frac{10}{3}.$$

Which means for every 3 steps to the right, we need to go 10 steps up.
(2) $(\underbrace{-6}_{x_1}, \underbrace{7}_{y_1})$ and $(\underbrace{0}_{x_2}, \underbrace{1}_{y_2})$.

$$m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{1 - 7)}{0 - (-6)} = \frac{-6}{6} = \frac{-1}{1}.$$

Which means for every one step to the right, we need to go one step down.

Exercise

Find the slope of the line that passes through

(1) $(5, 2)$ and $(4, -3)$.

(2) $(1, 7)$ and $(-9, 0)$.

(3) $(5, 2)$ and $(4, 2)$.

(4) $(3, 1)$ and $(3, 3)$.

2 - Equation of the line

To get the equation of a line, you need to find

- One point on the line (x_1, y_1) and
- The slope of the line m.

Then, the equation of the line is

$$y - y_1 = m(x - x_1) \qquad - - - \qquad \text{“point–slope form“}$$

Other forms:

General Linear Form $ax + by + c = 0$, where a, b, and c have **no** common factor.

Slope–Intercept Form $y = mx + b$, where m is the slope of the line and $(0, b)$ is the y–intercept.

Special Case: The equation of the vertical line is $x = x_1$.

Example

Find a general linear equation ($ax + by + c = 0$) of the line with the following properties:
(1) passes through $(1, -7)$ and has slope -3.

Solution:

$$\begin{aligned} y - y_1 &= m(x - x_1) \\ y - (-7) &= -3(x - 1) \\ y + 7 &= -3x + 3 \\ y + 3x + 7 - 3 &= 0 \\ y + 3x + 4 &= 0 \end{aligned}$$

Example

Find a general linear equation ($ax + by + c = 0$) of the line with the following properties:
(2) passes through $(-3, 4)$ and $(6, -4)$.

Solution: First we find the slope m which is

$$m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{-4 - 4}{6 - (-3)} = \frac{-8}{9}$$

$$\begin{aligned} y - y_1 &= m(x - x_1) \\ y - 4 &= \frac{-8}{9}(x - (-3)) \\ y - 4 &= \frac{-8}{9}(x + 3) \\ 9(y - 4) &= -8(x + 3) \\ 9y - 36 &= -8x - 24 \\ 9y + 8x - 36 + 24 &= 0 \\ 9y + 8x - 12 &= 0 \end{aligned}$$

Example

Find a general linear equation ($ax + by + c = 0$) of the line with the following properties:
(3) has slope 4 and y–intercept -4.

Solution:

$$y = mx + b$$
$$y = 4x - 4$$
$$y - 4x + 4 = 0$$

Example

Find a general linear equation ($ax + by + c = 0$) of the line with the following properties:
(4) is a vertical line passes through $(-2, -7)$.

Solution:

$$x = x_1$$
$$x = -2$$
$$x + 2 = 0$$

Exercise

Find a general linear form equation of the line that

(1) passes $(-5, 5)$ and has slope $\frac{-1}{3}$.

(2) passes $(2, 4)$ and $(-1, 3)$.

(3) has slope $\frac{-2}{3}$ and y–intercept -2.

(4) vertical line passes $(8, 0)$.

(5) Horizontal line passes $(2, 5)$.

Example

Find the slope and the y–intercept of a line with equation $y = 5(2 - 3x)$ and sketch its graph.

Solution: We need to write the equation in the slope–intercept form (i.e., we isloate y).

$$y = 5(2 - 3x)$$
$$y = 10 - 15x$$
$$y = -15x + 10$$

so we have

$$\text{Slope} = \frac{-15}{1} \text{ and } y\text{-intercept} = (0, 10)$$

To sketch the graph of the line, we start at the point $(0, 10)$ and for every one step to the right, we go 15 steps down.

Exercise

Find the slope and the y-intercept general linear form equation of the line that

(1) $2x = 5 - 3y$.

(2) $3(x-4) - 7(y+1) = 2$.

(3) $y = \frac{1}{2}x + 8$.

(4) $\frac{-x}{3} + \frac{3}{2}y = -1\frac{1}{2}$.

Example

Find a (a) general linear equation and (b) slope–intercept form of the line with an equation:
(1) $2x = 5 - 3y$.

Solution: (a) General Linear form:

$$2x = 5 - 3y$$
$$2x + 3y - 5 = 0$$

(b) Slope–intercept form:

$$2x = 5 - 3y$$
$$2x - 5 = -3y$$
$$\frac{2}{-3}x - \frac{5}{-3} = y$$

Example

Find a (a) general linear equation and (b) slope–intercept form of the line with an equation:
(2) $3(x-4)-7(y+1)=2$.

Solution: (a) General Linear form:

$$\begin{aligned} 3(x-4)-7(y+1) &= 2 \\ 3x-12-7y-7 &= 2 \\ 3x-7y-19-2 &= 0 \\ 3x-7y-21 &= 0 \end{aligned}$$

(b) Slope–intercept form:

$$\begin{aligned} 3x-7y-21 &= 0 \\ 3x-21 &= 7y \\ \frac{3}{7}x-\frac{21}{7} &= y \\ \frac{3}{7}x-3 &= y \end{aligned}$$

Example

Find a (a) general linear equation and (b) slope–intercept form of the line with an equation:
(3) $y = \frac{1}{2}x + 8$.

Solution: (a) General Linear form:

$$y = \frac{1}{2}x + 8$$
$$2y = x + 16$$
$$2y - x - 16 = 0$$

(b) Slope–intercept form:

$$y = \frac{1}{2}x + 8$$

Example

Find a (a) general linear equation and (b) slope–intercept form of the line with an equation:
(4) $\frac{-x}{3} + \frac{3}{2}y = -1\frac{1}{2}$.

Solution: (a) General Linear form:

$$\frac{-x}{3} + \frac{3}{2}y = -1\frac{1}{2}$$
$$-x + \frac{9}{2}y = -3\frac{1}{2}$$
$$-2x + 9y = -9$$
$$-2x + 9y + 9 = 0$$

(b) Slope–intercept form:

$$-2x + 9y + 9 = 0$$
$$9y = 2x - 9$$
$$y = \frac{2}{9}x - \frac{9}{9}$$

3 - Parallel and Perpendicular Lines

Definition

- Two lines are parallel if

$$m_1 = m_2$$

- Two lines are perpendicular if

$$m_1 m_2 = -1$$

Example

Determine whether the given lines are parallel, perpendicular, or neither?
(1) $y = -5x + 7$ and $y = -5x - 2$.

Solution:

$$m_1 = -5 \text{ and } m_2 = -5$$

So the two lines are parallel.

Example

Determine whether the given lines are parallel, perpendicular, or neither?
(2) $x + 3y + 5 = 0$ and $y = 3x$.

Solution:

$$3y = -x + 5 \text{ and } y = 3x$$
$$3y = \frac{-1}{3}x + \frac{5}{3} \text{ and } y = 3x$$
$$m_1 = \frac{-1}{3} \text{ and } m_2 = 3$$

Now $m_1 m_2 = (\frac{-1}{3})3 = -1$. Thus the two lines are perpendicular lines.

Example

Determine whether the given lines are parallel, perpendicular, or neither?
(2) $x - 2 = 3$ and $y = 2$.

Solution:

$$\begin{aligned} x - 2 = 3 &\text{ and } y = 2 \\ x = 5 &\text{ and } y = 2 \\ m_1 = \text{undefined} &\text{ and } m_2 = 0 \end{aligned}$$

Note that the line without slope is a vertical line. The line with a slope of zero is a horizontal line. Thus the two lines are perpendicular lines.

Exercise

Determine whether the given lines are parallel, perpendicular, or neither?
(1) $x + 2y = 0$ and $x + 4y - 7 = 0$.
(2) $x = 3$ and $x = -2$.
(3) $y = 4x + 7$ and $4x - y + 6 = 0$.

Example

Find an equation of the line

(1) Passes $(2, 1)$ and parallel to the line $y - 4x + 6 = 0$.

Solution: We need to find the slope first, since both lines are parallel, they have the same slope, so we have

$$m_1 = m_2 = 4$$

$$y - y_1 = m(x - x_1)$$
$$y - 1 = 4(x - 1)$$
$$y = 4x - 3$$

Example

Find an equation of the line

(2) Passes $(3, 3)$ and perpendicular to the line $2x + 3y - 2 = 0$.

Solution: We need to find the slope first, since both lines are perpendicular, then

$$m_1 m_2 = -1$$

We need to find m_2 to find m_1. In order to find m_2, we solve the equation for y.

$$\begin{aligned} 2x + 3y - 2 &= 0 \\ 3y &= -2x + 2 \\ y &= \frac{-2}{3}x + \frac{2}{3} \end{aligned}$$

Example

Find an equation of the line

(2) Passes $(3,3)$ and perpendicular to the line $2x+3y-2=0$.

Solution: So $m=\frac{-2}{3}$. Thus we have

$$m_1 m_2 = -1$$
$$\frac{-2}{3} m_1 = -1$$
$$m_1 = \frac{3}{2}$$

So the equation of the line is

$$y - y_1 = m(x - x_1)$$
$$y - 3 = \frac{3}{2}(x - 3)$$
$$2y - 6 = 3x - 9$$
$$2y - 3x + 3 = 0$$

Exercise

Find the equation of the line that

(1) parallel to $2x + 3y + 6 = 0$ and passes $(-7, -9)$.

(2) Perpendicular to $3y = \frac{-5}{2}x + 7$ and passes $(4, -1)$.

Exercise

(Old Exam Question)
(1) Find the slope and the y-intercept of $3x + 6y - 12 = 0$.
(2) Find an equation of the line passing through $(1, -2)$ and parallel to $2y = 5 - 8x$.
(3) Find an equation of the line passing through $(2, 1)$ and perpendicular to $3y + x = 18$.