Section 4.1
Exponential Functions

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MATHS 103: Mathematics for Business I

Topics

1. Exponential Function, graph, and its properties.
2. Compound Interest (Application to Finance).
3. The Euler Number e.

1- The Exponential Function

Definition

The function

$$f(x) = a^x, \qquad a > 0, a \neq 1$$

is called an **exponential function**. The number a is called the **base** and x is called the **exponent** (**power**).

Recall: Rule for the exponent

1. $a^x \cdot a^y = a^{x+y}$.
2. $\frac{a^x}{a^y} = a^{x-y}$.
3. $(a^x)^y = a^{xy}$.
4. $(ab)^x = a^x b^y$.
5. $a^{-x} = \frac{1}{a^x}$.
6. $a^0 = 1$.
7. $a^1 = a$.

Graphing Exponential Function with $a > 1$

Example

Graph the function

$$f(x) = 3^x$$

Solution: Using the calculator, we fill the following table

x	-2	-1	0	1	2	3
y						

- Domain $= (-\infty, \infty)$.
- Co–domain$=(-\infty, \infty)$.
- Range$=(0, \infty)$.
- y–intercept $= (0, 1)$.
- x–intercept $=$ none.

Exercise

Graph $f(x) = 7^x$ and observe the difference with the previous example.

Graphing Exponential Function with $0 < a < 1$

Example

Graph the function

$$f(x) = \left(\frac{1}{3}\right)^x$$

Solution:Using the calculator, we fill the following table

x	-2	-1	0	1	2	3
y						

- Domain $= (-\infty, \infty)$.
- Co–domain$=(-\infty, \infty)$.
- Range$=(0, \infty)$.
- y–intercept $= (0, 1)$.
- x–intercept $=$ none.

Exercise

Graph $f(x) = \left(\frac{1}{7}\right)^x$ and observe the difference with the previous example.

Summary

$$y = f(x) = a^x$$

- Domain $= (-\infty, \infty)$.
- Co–domain$=(-\infty, \infty)$.
- Range$=(0, \infty)$.
- y–intercept $= (0, 1)$.
- x–intercept $=$ none.

Graphing Exponential Function with $a > 1$

Example

Graph the function

$$f(x) = 3^{x+1} - 2$$

Solution:Using the calculator, we fill the following table

x	-3	-2	-1	0	1	2	3
y							

- Domain $= (-\infty, \infty)$.
- Co–domain$=(-\infty, \infty)$.
- Range$=(-2, \infty)$.
- y–intercept $= (0, 1)$.
- x–intercept $=$ later in Section 4.4.

2 - Compound Interest

Example

Suppose you save 100 BD in a saving account that pays 1% annually. Find the total money in your account every year.

Solution: Let A_n be the amount in the account in year n, i.e., A_2 is the amount in the account after 2 years.

- Year 0: $A_0 = 100$ BD.
- Year 1: $A_1 = 100 + 100(0.01) = 101$ BD.
  Year 1: $A_1 = A_0 + A_0(r) = A_0(1 + r)$.
- Year 2: $A_2 = 101 + 101(0.01) = 102.1$ BD.
  Year 2:
  $A_2 = A_1 + A_1(r) = A_1(1 + r) = A_0(1 + r)(1 + r) = A_0(1 + r)^2$.
- Year 3: $A_3 = 102.1 + 102.1(0.01) = 103.03$ BD.
  Year 3: $A_3 = A_2 + A_2(r) = A_2(1 + r) = A_0(1 + r)^3$.
- Year n: $A_n = A_{n-1} + A_{n-1}(r) = A_{n-1}(1 + r) = A_0(1 + r)^n$.

So in any year n, we have

$$A_n = P(1 + r)^n$$

Example

Suppose you saved 3000 BD at 5% for 3 years. Find the compound amount and the compound interest.

Solution: $n = 3$, $P = 3000$, and $r = 5\% = 0.05$. We have that
$A_3 = P(1 + r)^n = 3000(1 + 0.05)^3 = 3472.875$ BD.
The total interest is $I = A_3 - P = 3472.875 - 3000 = 472.875$ BD.

The compound interest formula

In general, if the interest are given periodically (say m times a year), the formula is

$$A_n = P\left(1 + \frac{r}{m}\right)^{mn}$$

Example

Find the compund interest and the compound interest of (a) 500 BD for 7 years at 11% semi–annually

Solution: $n = 7$, $P = 500$, $r = 11\% = 0.11$, and $m = 2$. We have that $A_7 = P(1 + \frac{r}{m})^{nm}$=$500(1 + \frac{0.11}{2})^{3\cdot 2}$= 1658.04 BD.
The total interest is $I = A_7 - P = 1658.04 - 500 = 1158.04$ BD.

Example

Find the compund interest and the compound interest of (b) 4000 BD for 15 years at 8.5% quarterly

Solution: $n = 15$, $P = 4000$, $r = 8.5\% = 0.085$, and $m = 4$. We have that $A_{15} = P(1 + \frac{r}{m})^{nm}$=$4000(1 + \frac{0.085}{4})^{15\cdot 4}$= 14124.86 BD.
The total interest is $I = A_{15} - P = 14124.86 - 4000 = 10124.86$ BD.

Exercise

Find the compound amount and the compound interest of investing

1. 300 BD at 7% for 9 years compounded yearly.
2. 200 BD at 5% for 6 years compounded monthly.
3. 1000 BD at 9% for 2 years compounded semi–annually.
4. 200 BD at 1% for 2 years compounded daily (365 days in one year).
5. (Old exam question) 11000 BD at 3% for 9 years compounded monthly.
6. (Old exam question) 1020 BD at 6% for 8 years compounded monthly.

3 - The Euler Number

Example

(Motivational Example) Suppose you invest 1 BD in an account that pays 100%. Find the compound amount for one year in every possible period. What happen if the interest are paid continuously in every single moment?

Solution: $P = 1$, $r = 100\% = 1$, $n = 1$, $m = m$. So the compound amount is

$$A_1 = P(1 + \frac{r}{m})^{nm} = 1(1 + \frac{1}{m})^m$$

Continue...

$$A_1 = (1 + \frac{1}{m})^m$$

Period	m	A_m
yearly	1	
semi–annually	2	
quarterly	4	
monthly	12	
daily	365	
hourly	365(24)	
Minutely	365(24)(60)	
secondly	365(24)(60)(60)	
mini–secondly	$365(24)(60)(60)(10)^3$	
micro–secondly	$365(24)(60)(60)(10)^6$	
nano–secondly	$365(24)(60)(60)(10)^9$	
Continuously	∞	

As you can see from the example above that as $m \to \infty$, $A_1 \to 2.718281828\ldots$. We define

$$2.718281828\cdots = \lim_{m\to\infty}(1+\frac{1}{m})^m = e$$

- e is called the **Euler number**.
- e is not a rational number,i.e., the decimal expansion of e never ends nor repeat in a pattern.
- $$e = 1 + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \frac{1}{4!} + \frac{1}{5!} + \ldots$$

Exercise

Find the value of (a) $e^{2.5}$ (b) e^{-1} (c) $e^{\frac{1}{3}}$.

Graphing Exponential Function with $a > 1$

Example

Graph the function
$$f(x) = -e^{-x+3}$$

Solution:Using the calculator, we fill the following table

x	-2	-1	0	1	2	3
y						