Section 4.3
Properties of Logarithms

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MATHS 103: Mathematics for Business I

Properties

1. $\log_a(m \cdot n) = \log_a m + \log_a n$.
2. $\log_a(\frac{m}{n}) = \log_a m - \log_a n$.
3. $\log_a m^r = r \log_a m$.
4. $\log_a 1 = 0$.
5. $\log_a a = 1$.
6. (change of bases) $\log_a m = \frac{\log_b m}{\log_b a}$.

Proof (Not required)

First recall the fundamental equations: (The fundamental equations)

1. $a^{\log_a x} = x.$
2. $\log_a a^x = x.$

1- We want to prove that

$$\log_a(mn) = \log_a m + \log_a n$$

which is the same as proving the exponential form of the above which is

$$mn = a^{\log_a m + \log_a n}$$

RHS$= a^{\log_a m + \log_a n} = a^{\log_a m} a^{\log_a n} = mn =$LHS.
2- (exercise).
3- (exercise).

Proof (Not required)

4- We want to prove that $\log_a 1 = 0$. Let $\log_a 1 = x$. We have in the exponential form that $1 = a^x$, which is $a^0 = a^x$ and so $x = 0$.
5- We want to prove that $\log_a a = 1$. Let $\log_a a = x$. We have in the exponential form that $a = a^x$,so $x = 1$.
6- We want to prove that

$$\log_a m = \frac{\log_b m}{\log_b a}$$

which is the same as proving

$$\log_a m \cdot \log_b a = \log_b m$$

which is the same as proving the exponential form of the above which is

$$b^{\log_a m \log_b a} = m$$

LHS= $b^{\log_a m \log_b a} = \left(b^{\log_b a}\right)^{\log_a m} = a^{\log_a m} = m =$ RHS.

Example

Let $\log 2 = a, \log 3 = b$, and $\log 5 = c$. Find in terms of a, b, and c the following

1. $\log 6 = \log(2 \cdot 3) = \log 2 + \log 3 = a + b$.
2. $\log 15 = \log(3 \cdot 5) = \log 3 + \log 5 = b + c$.
3. $\log 60 = \log(2^2 \cdot 3 \cdot 5) = \log 2^2 + \log 3 + \log 5 = 2\log 2 + \log 3 + \log 5 = 2a + b + c$.
4. $\log_2 3 = \frac{\log 3}{\log 2} = \frac{b}{c}$.
5. $\log 1000 = \log(2^3 \cdot 5^3) = \log 2^3 + \log 5^3 = 3\log 2 + 3\log 5 = 3a + 3c$.

Exercise

In the previous exercise, find
(1) $\log_5 3$ (2) $\log 10$ (3) $\log 0.00002$. (4) $\log \frac{25}{6}$.

Exercise

If $\log_a 5 = 0.83$ and $\log_a 3 = 0.56$. Find
(1) $\log_a 15$ (2) $\log_a 25$ (3) $\log_a(\sqrt{3})$.

Example

(Expansion) Write the following expression as sum or difference of logarithms

1. $\ln(\frac{x}{wz^2}) = \ln x - \ln(wz^2) = \ln x - (\ln w + \ln z^2) = \ln x - \ln w - 2\ln z.$
2. $\ln(\frac{x+1}{x+5})^4 = 4\ln(\frac{x+1}{x+5}) = 4(\ln(x+1) - \ln(x+5)).$
3. $\ln(\frac{\sqrt{x}}{(x^2)(x+3)^4}) = \ln\sqrt{x} - \ln x^2 - \ln(x+3)^4 =$
   $\ln x^{\frac{1}{2}} - 2\ln x - 4\ln(x+3) = \frac{1}{2}\ln x - 2\ln x - 4\ln(x+3) =$
   $-\frac{3}{2}\ln x - 4\ln(x+3).$

Exercise

Write each of the following expression as sum or difference of logarithms:
(1) $\log_3(\frac{5\cdot 7}{4})$ (2) $\log_2(\frac{x^5}{y^2})$ (3) $\log(\frac{x^2 z}{wy^2})$.

Example

(Single Logarithm) Write each of the following as a single logarithm.

1. $\log 6 + \log 4 = \log(6 \cdot 4) = \log 24$.
2. $2 \log x - \frac{1}{2} \log(x-3) = \log x^2 - \log(x-3)^{\frac{1}{2}} = \log \frac{x^2}{(x-3)^{\frac{1}{2}}}$.
3. $2 + 10 \log 3 = 2 \log 10 + 10 \log 3 = \log 10^2 + \log 3^{10} = \log(10^2 \cdot 3^{10})$.

Exercise

Write each of the following as a single logarithm.

1. $2 \log_5 3 + 3 \log_5 2$.
2. $3 \log_a x - \log_a(x+1)$.
3. $\log_4 25 + \log_4 3 - \log_4 5$.
4. $\log_5 8 - \log_5 x$.
5. $\log_{10} 27 - \log 3$.
6. $\log_3(x^2+5) - \log_3(x^2+1)$.

First recall the fundamental equations: (The fundamental equations)

1. $a^{\log_a x} = x$.
2. $\log_a a^x = x$.

Example

Find the value of the following:

1. $\log_5 5^{212} = 212 \log_5 5 = 212$.
2. $\ln e^{0.1} = 0.1 \ln e = 0.1$.
3. $\log \frac{1}{10} + \ln e^3 = \log 10^{-3} + \ln e^3 = -3 \log 10 + 3 \ln e = -3 + 3 = 0$.
4. $e^{\ln 5} = 5$ (by the fundamental equation).