Section 6.4
Solving Linear Systems by Row Operations

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MATHS 103: Mathematics for Business I

Goal: To solve a system of linear equations by using *elementary row operations* on matrices.

What are the *elementary row operations* on a matrix?

1. Interchanging any two rows ($R_i \leftrightarrow R_j$).
2. Multiplying (dividing) a row by a non–zero number ($R_i \to cR_i$).
3. Add a multiple of a row to another row ($R_i \to R_i + cR_j$).

Example

Consider the following matrix

$$\begin{pmatrix} 1 & 0 & 2 \\ 2 & 5 & 1 \\ 3 & 0 & -2 \end{pmatrix}$$

Perform $R_3 \to R_3 + 2R_1$.

Solution:

$$\begin{pmatrix} 1 & 0 & 2 \\ 2 & 5 & 1 \\ 3+2(1) & 0+2(0) & -2+2(2) \end{pmatrix} = \begin{pmatrix} 1 & 0 & 2 \\ 2 & 5 & 1 \\ 5 & 0 & 2 \end{pmatrix}$$

Goal of the elementary row operations:

We want to reach a **reduced** matrix, which is a matrix that satisfy the following properties:

1. All zero–rows are at the bottom of the matrix.
2. Each non–zero row has a leading 1's (called **pivot**) and all entries in the pivot columns are zeros.
3. The pivots start from left to right (up to down).

Example

Which of the following matrices are reduced matrix?

$$\begin{pmatrix} 1 & 3 & 0 & 5 & 1 \\ 0 & 0 & 1 & 2 & 6 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$$

Example

Reduce the matrix

$$\begin{pmatrix} 0 & -3 & 0 & 2 \\ 1 & 5 & 0 & 2 \end{pmatrix}$$

Solution:

$$\begin{pmatrix} 0 & -3 & 0 & 2 \\ 1 & 5 & 0 & 2 \end{pmatrix}, \qquad R_1 \leftrightarrow R_2$$

$$\begin{pmatrix} 1 & 5 & 0 & 2 \\ 0 & -3 & 0 & 2 \end{pmatrix}, \qquad R_2 \to \frac{1}{-3} R_2$$

$$\begin{pmatrix} 1 & 5 & 0 & 2 \\ 0 & 1 & 0 & \frac{2}{-3} \end{pmatrix}, \qquad R_1 \to R_1 - 5R_2$$

$$\begin{pmatrix} 1-5(0) & 5-5(1) & 0-5(0) & 2-5(\frac{2}{-3}) \\ 0 & 1 & 0 & \frac{2}{-3} \end{pmatrix}$$

$$\begin{pmatrix} 1 & 0 & 0 & \frac{16}{3} \\ 0 & 1 & 0 & \frac{2}{-3} \end{pmatrix}$$

Solving System of Linear Equations using elementary row operations

Example

Solve the system

$$2x - 7y = -1$$
$$x + 3y = 6$$

Solution: First we write the **augmented matrix** of the system which is

$$\left(\begin{array}{cc|c} 2 & -7 & -1 \\ 1 & 3 & 6 \end{array}\right), \qquad R_1 \leftrightarrow R_2$$

$$\left(\begin{array}{cc|c} 1 & 3 & 6 \\ 2 & -7 & -1 \end{array}\right), \qquad R_2 \to R_2 - 2R_1$$

$$\left(\begin{array}{cc|c} 1 & 3 & 6 \\ 2-2(1) & -7-2(3) & -1-2(6) \end{array}\right)$$

$$\begin{pmatrix} 1 & 3 & | & 6 \\ 2-2(1) & -7-2(3) & | & -1-2(6) \end{pmatrix},$$

$$\begin{pmatrix} 1 & 3 & | & 6 \\ 0 & -13 & | & -13 \end{pmatrix} \qquad R_2 \to \frac{1}{-13} R_2$$

$$\begin{pmatrix} 1 & 3 & | & 6 \\ 0 & 1 & | & 1 \end{pmatrix} R_1 \to R_1 - 3R_2$$

$$\begin{pmatrix} 1-3(0) & 3-3(1) & | & 6-3(1) \\ 0 & 1 & | & 1 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 0 & | & 3 \\ 0 & 1 & | & 1 \end{pmatrix}$$

So $x = 3$ and $y = 1$ and thus the solution set is $\{(3, 1)\}$

Example

Solve the system

$$\begin{aligned} x + 4y &= 9 \\ 3x - y &= 6 \\ 2x - 2y &= 4 \end{aligned}$$

Solution: First we write the **augmented matrix** of the system which is

$$\left(\begin{array}{cc|c} 1 & 4 & 9 \\ 3 & -1 & 6 \\ 2 & -2 & 4 \end{array}\right), \qquad R_2 \to R_2 - 3R_1 \quad R_3 \to R_3 - 2R_1$$

$$\left(\begin{array}{cc|c} 1 & 4 & 9 \\ 3-3(1) & -1-3(4) & 6-3(9) \\ 2-2(1) & -2-2(4) & 4-2(9) \end{array}\right)$$

$$\left(\begin{array}{cc|c} 1 & 4 & 9 \\ 0 & -13 & -21 \\ 0 & -10 & -14 \end{array}\right) \qquad R_2 \to \frac{1}{-13}R_2$$

$$\begin{pmatrix} 1 & 4 & | & 9 \\ 0 & 1 & | & \frac{-21}{-13} \\ 0 & -10 & | & -14 \end{pmatrix}, \qquad R_3 \to R_3 + 10R_2$$

$$\begin{pmatrix} 1 & 4 & | & 9 \\ 0 & 1 & | & \frac{-21}{-13} \\ 0 & -10 + 10(1) & | & -14 + 10(\frac{-21}{-13}) \end{pmatrix}$$

$$\begin{pmatrix} 1 & 4 & | & 9 \\ 0 & 1 & | & \frac{-21}{-13} \\ 0 & 0 & | & \frac{28}{13} \end{pmatrix}$$

We have $0 = \frac{28}{13}$ which is a false statement and thus there will be no solution.

Example

Solve the system

$$\begin{aligned} x + y - z &= 7 \\ 4x + 6y - 4z &= 8 \\ x - y - 5z &= 23 \end{aligned}$$

Solution: First we write the **augmented matrix** of the system which is

$$\left(\begin{array}{ccc|c} 1 & 1 & -1 & 7 \\ 4 & 6 & -4 & 8 \\ 1 & -1 & -5 & 23 \end{array}\right), \quad R_2 \to R_2 - 4R_1 \quad R_3 \to R_3 - R_1$$

$$\left(\begin{array}{ccc|c} 1 & 1 & -1 & 7 \\ 4-4(1) & 6-4(1) & -4-4(-1) & 8-4(7) \\ 1-1 & -1-1 & -5-(-1) & 23-7 \end{array}\right)$$

$$\left(\begin{array}{ccc|c} 1 & 1 & -1 & 7 \\ 0 & 2 & 0 & -20 \\ 0 & -2 & -4 & 16 \end{array}\right) \qquad R_2 \to \frac{1}{2}R_2$$

$$\begin{pmatrix} 1 & 1 & -1 & | & 7 \\ 0 & 1 & 0 & | & -10 \\ 0 & -2 & -4 & | & 16 \end{pmatrix}, \qquad R_3 \to R_3 + 2R_2$$

$$\begin{pmatrix} 1 & 1 & -1 & | & 7 \\ 0 & 1 & 0 & | & -10 \\ 0 & -2+2(1) & -4+2(0) & | & 16+2(-10) \end{pmatrix}$$

$$\begin{pmatrix} 1 & 1 & -1 & | & 7 \\ 0 & 1 & 0 & | & -10 \\ 0 & 0 & -4 & | & -4 \end{pmatrix} R_3 \to \frac{1}{-4} R_3$$

$$\begin{pmatrix} 1 & 1 & -1 & | & 7 \\ 0 & 1 & 0 & | & -10 \\ 0 & 0 & 1 & | & 1 \end{pmatrix} \qquad R_1 \to R_1 + R_3$$

$$\begin{pmatrix} 1+1(0) & 1+1(0) & -1+1(1) & | & 7+1(1) \\ 0 & 1 & 0 & | & -10 \\ 0 & 0 & 1 & | & 1 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 1 & 0 & | & 8 \\ 0 & 1 & 0 & | & -10 \end{pmatrix} \qquad R_1 \to R_1 - R_2$$

$$\begin{pmatrix} 1 & 1 & 0 & | & 8 \\ 0 & 1 & 0 & | & -10 \\ 0 & 0 & 1 & | & 1 \end{pmatrix} \qquad R_1 \to R_1 - R_3$$

$$\begin{pmatrix} 1 & 0 & 0 & | & 18 \\ 0 & 1 & 0 & | & -10 \\ 0 & 0 & 1 & | & 1 \end{pmatrix}$$

So $x = 18$, $y = -10$, and $z = 1$.
Solution Set $= \{(18, -10, 1)\}$.

Example

Solve the system

$$\begin{aligned} x + 3y &= 2 \\ 2x + 7y &= 4 \\ 3x + 15y + 3z &= 15 \end{aligned}$$

Solution: First we write the **augmented matrix** of the system which is

$$\left(\begin{array}{ccc|c} 1 & 3 & 0 & 2 \\ 2 & 7 & 0 & 4 \\ 3 & 15 & 3 & 15 \end{array}\right), \qquad R_2 \to R_2 - 2R_1 \quad R_3 \to R_3 - 3R_1$$

$$\left(\begin{array}{ccc|c} 1 & 3 & 0 & 2 \\ 2-2(1) & 7-2(3) & 0-2(0) & 4-2(2) \\ 3-3(1) & 15-3(3) & 3-3(0) & 15-3(2) \end{array}\right)$$

$$\left(\begin{array}{ccc|c} 1 & 3 & 0 & 2 \\ 0 & 1 & 0 & 0 \\ 0 & 6 & 3 & 9 \end{array}\right) \qquad R_3 \to R_3 - 6R_2$$

$$\begin{pmatrix} 1 & 3 & 0 & | & 2 \\ 0 & 1 & 0 & | & 0 \\ 0-6(0) & 6-6(1) & 3-6(0) & | & 9-6(0) \end{pmatrix}$$

$$\begin{pmatrix} 1 & 3 & 0 & | & 2 \\ 0 & 1 & 0 & | & 0 \\ 0 & 0 & 3 & | & 9 \end{pmatrix} \qquad R_3 \to \frac{1}{3} R_3$$

$$\begin{pmatrix} 1 & 3 & 0 & | & 2 \\ 0 & 1 & 0 & | & 0 \\ 0 & 0 & 1 & | & 3 \end{pmatrix} R_1 \to R_1 - 3R_2$$

$$\begin{pmatrix} 1-3(0) & 3-3(1) & 0-3(0) & | & 2-3(0) \\ 0 & 1 & 0 & | & 0 \\ 0 & 0 & 1 & | & 3 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 0 & 0 & | & 2 \\ 0 & 1 & 0 & | & 0 \\ 0 & 0 & 1 & | & 3 \end{pmatrix}$$

Example

(Old Exam Question) The supply and demand equations of a certain product are

$$\text{Demand}: 2q + p = 50$$
$$\text{Supply}: 3q - 5p = 10$$

Use *elementary row operations* to find the market equilibrium point.

Solution: First we write the **augmented matrix** of the system which is

$$\left(\begin{array}{cc|c} 2 & 1 & 50 \\ 3 & -5 & 10 \end{array}\right), \qquad R_1 \to \frac{1}{2} R_1$$

$$\left(\begin{array}{cc|c} 1 & \frac{1}{2} & 25 \\ 3 & -5 & 10 \end{array}\right), \qquad R_2 \to R_2 - 3R_1$$

$$\left(\begin{array}{cc|c} 1 & \frac{1}{2} & 25 \\ 3 - 3(1) & -5 - 3(\frac{1}{2}) & 10 - 3(25) \end{array}\right)$$

$$\begin{pmatrix} 1 & \frac{1}{2} & | & 25 \\ 0 & -\frac{13}{2} & | & -65 \end{pmatrix}, \qquad R_2 \to \frac{1}{\frac{-13}{2}} R_2$$

$$\begin{pmatrix} 1 & \frac{1}{2} & | & 25 \\ 0 & 1 & | & 10 \end{pmatrix} \qquad R_1 \to R_1 - \frac{1}{2} R_2$$

$$\begin{pmatrix} 1 - \frac{1}{2}(0) & \frac{1}{2} - \frac{1}{2}(1) & | & 25 - \frac{1}{2}(10) \\ 0 & 1 & | & 10 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 0 & | & 20 \\ 0 & 1 & | & 10 \end{pmatrix}$$

So $q = 20$, $p = 10$.