University of Bahrain
Department of Mathematics
MATHS311: Abstract Algebra 1
Fall 2017
Dr. Abdulla Eid

Homework 1: Examples of Groups
Due on October 12
Hand on 1,2,3,5, and (4 or 6)

Name: ______________________________

1. Let $\mathbb{Q}^{\times}$ be the set of non–zero rational numbers. Define a binary operation $*$on $\mathbb{Q}^{\times}$ by

$$a * b := a + b + ab.$$

Verify whether $(\mathbb{Q}^{\times}, *)$ is a group or not.

2. Define an operation $\circ$ on the set $\mathbb{Z}$ by

$$a \circ b := a + b + 1$$

Prove that $(\mathbb{Z}, \circ)$ is an abelian group.

3. Write down the cayley's table for the following groups:

   1. $(\mathbb{Z}_7, \cdot_7)$.

   2. $(U(12), \cdot_{12})$.

3. $(\mathbb{Z}_4, +_4)$ with identification of $e = 0, a = 1, b = 2, c = 3$.

4. Assume $(\{e, a, b, c\}, *)$ is a group different than the one above. What is the Cayley's table of this group? (it is called the Klein-four group)

4. Let X be a non–empty set. The power set of X is the set of all subsets of X, denoted by $\mathbb{P}(X)$. Define a binary operation Δ on $\mathbb{P}(X)$ by

$$A\Delta B := (A \cup B) - (A \cap B)$$

   Show that $(\mathbb{P}(X), \Delta)$ is an abelian group. (You do not need to check the associativity for now).

5. In this exercise, we want to show that $\mathrm{Sym}(X)$ is a group under the composition of functions.

   1. Let $f, g : X \to X$ be two bijective functions. Prove that $g \circ f : X \to X$ is again a bijective function.

   2. Show that $(h \circ g) \circ f = h \circ (g \circ f)$ for all $f, g, h \in \mathrm{Sym}(X)$.

   3. Show that the identity function defined by

$$\begin{aligned} id_X : X &\to X \\ x &\mapsto x \end{aligned}$$

   is bijective and such that we have $f \circ id_X = f$.

4. Prove that the function $f^{-1} : X \to X$ is bijective and such that $f \circ f^{-1} = id_X$

6. (Heisenberg group) Prove that the set of 3×3 matrices with real entries of the form

$$\begin{pmatrix} 1 & x & y \\ 0 & 1 & z \\ 0 & 0 & 1 \end{pmatrix}$$

is a group under the matrix multiplication. This group, known as the *Heisenberg* group, is important in quantum physics and in particular in Heisenberg uncertainty principle.

(Note: Do not show the associativity, but you will need to prove that the Heisenberg group is closed under matrix multiplication).