University of Bahrain
Department of Mathematics
MATHS311: Abstract Algebra 1
Fall 2017
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Project 4: Monoid into Groups

The aim of this project is to provide further practice in:

1. Monoids
2. Groups
3. Equivalence classes
4. Well defined operations
5. Group monomorphisms
6. Grothendieck Construction

In this project, we will us the Grothendieck construction to turn the monoid $(\mathbb{N}, +)$ into an abelian group (what do you think is the natural generalization of $(\mathbb{N}, +)$?)

Consider the set $G' = \mathbb{N} \times \mathbb{N} = \{(a, b) \mid a, b \in \mathbb{N}\}$. Define a relation on G' by

$$(a, b) \sim (c, d) :\iff a + d = b + c$$

1. Show that $\sim$ is an equivalence relation on G'.

2. What is the equivalence classes of $(0, 0)$, $(1, 1)$, $(2, 2)$, $(3, 1)$, $(1, 3)$, $(0, 10)$, $(0, 9)$, $(10, 0)$? can you in general describe the elements in the equivalence class of any (a, b)?

3. Let $G = G'/\sim$ be the set of all equivalence classes. Define an operation $\oplus$ on G by

   $$[(a, b)] \oplus [(c, d)] := [(a + c, b + d)]$$

   Show that this operation is well defined.

4. Prove that $(G, \oplus)$ is an abelian group.

5. Show that the map $i : \mathbb{N} \to G$ given by $i(a) := [(a, 0)]$ is a monomorphism.

6. Show that $(\mathbb{Z}, +)$ and $(G, \oplus)$ are isomorphic.
(Hint: If n is positive, then n should be associated with $[(n, 0)]$ and if n is negative, it should be associated with $[(0, n)]$)

7. Now can you generalize all the steps above using any monoid $(M, +)$? What is the condition on M for the above construction to work?