Section 1.3 (Part 3)
Matrix Multiplication

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MATHS 211: Linear Algebra

Goal: To define a multiplication between two matrices A and B.

When this is possible?

This is possible if the number of columns of A is the same as the number of rows of B.

$$A_{m\times n} \cdot B_{n\times k} = AB_{m\times k}$$

Example 1

Multiply

$$\begin{pmatrix}1 & 3 & 2\end{pmatrix} \cdot \begin{pmatrix}1\\1\\3\end{pmatrix}$$

Solution:

$$\begin{matrix} & \begin{pmatrix}1\\1\\3\end{pmatrix} \\ \begin{pmatrix}1 & 3 & 2\end{pmatrix} & \big(1(1)+3(1)+2(3)\big) \\ & =\big(10\big)\end{matrix}$$

Example 2

Multiply

$$\begin{pmatrix} 1 & 2 \\ -3 & 4 \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 \\ 5 & 6 \end{pmatrix}$$

Solution:

$$\begin{pmatrix} 1 & & 0 \\ 5 & & 6 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 2 \\ -3 & 4 \end{pmatrix} \begin{pmatrix} 1(1) + 2(5) & 1(0) + 2(6) \\ -3(1) + 4(5) & -3(0) + 4(6) \end{pmatrix}$$

$$= \begin{pmatrix} 11 & 12 \\ 17 & 24 \end{pmatrix}$$

Example 3

Multiply

$$\begin{pmatrix} 1 & 0 \\ 5 & 6 \end{pmatrix} \cdot \begin{pmatrix} 1 & 2 \\ -3 & 4 \end{pmatrix}$$

Solution:

$$\begin{pmatrix} 1 & 2 \\ -3 & 4 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 0 \\ 5 & 6 \end{pmatrix} \begin{pmatrix} 1(1) + 0(-3) & 1(2) + 0(4) \\ 5(1) + 6(-3) & 5(2) + 6(4) \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 2 \\ -13 & 34 \end{pmatrix}$$

Note: From the previous two examples, we have $AB \neq BA$.

Example 4

Multiply

$$\begin{pmatrix} 5 & 7 & 2 \\ 1 & 2 & -3 \end{pmatrix} \cdot \begin{pmatrix} 2 & 1 \\ -3 & -2 \\ 5 & 0 \end{pmatrix}$$

Solution:

$$\begin{matrix} & \begin{pmatrix} 2 & & 1 \\ -3 & & -2 \\ 5 & & 0 \end{pmatrix} \\ \begin{pmatrix} 5 & 7 & 2 \\ 1 & 2 & -3 \end{pmatrix} & \begin{pmatrix} 5(2) + 7(-3) + 2(5) & \\ & \end{pmatrix} \end{matrix}$$

continue...

$$
\begin{pmatrix} 2 & 1 \\ -3 & -2 \\ 5 & 0 \end{pmatrix}
$$

$$
\begin{pmatrix} 5 & 7 & 2 \\ 1 & 2 & -3 \end{pmatrix}
\begin{pmatrix} 5(2)+7(-3)+2(5) & 5(1)+7(-2)+2(0) \\ 1(2)+2(-3)+(-3)(5) & 1(1)+2(-2)+(-3)(0) \end{pmatrix}
$$

$$
= \begin{pmatrix} -1 & -9 \\ -20 & -3 \end{pmatrix}
$$

Example 5

Let

$$A = \begin{pmatrix} -1 & -1 & 0 \\ 0 & 1 & 1 \end{pmatrix}$$

Find AA^T

Solution:

$$\begin{matrix} & \begin{pmatrix} -1 & 0 \\ -1 & 1 \\ 0 & 1 \end{pmatrix} \\ \begin{pmatrix} -1 & -1 & 0 \\ 0 & 1 & 1 \end{pmatrix} & \begin{pmatrix} -1(-1) + -1(-1) + 0(0) & -1(0) + -1(1) + 0(1) \\ 0(-1) + 1(-1) + (1)(0) & 0(0) + 1(1) + 1(1) \end{pmatrix} \end{matrix}$$

$$= \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}$$

Example 6

Let

$$A = \begin{pmatrix} -1 & -1 & 0 \\ 0 & 1 & 1 \end{pmatrix}$$

Find A^2

Solution:

$$\begin{pmatrix} -1 & -1 & 0 \\ 0 & 1 & 1 \end{pmatrix}$$

$$\begin{pmatrix} -1 & -1 & 0 \\ 0 & 1 & 1 \end{pmatrix}$$

Which is not possible. Why?

Example 7

Multiply

$$\begin{pmatrix}1 & 2\\ 1 & 2\end{pmatrix}\cdot\begin{pmatrix}2 & -3\\ -1 & \frac{3}{2}\end{pmatrix}$$

Solution:

$$\begin{array}{cc} & \begin{pmatrix}2 & & -3\\ -1 & & \frac{3}{2}\end{pmatrix}\\ \begin{pmatrix}1 & 2\\ 1 & 2\end{pmatrix} & \begin{pmatrix}1(2)+2(-1) & 1(-3)+2(\frac{3}{2})\\ 1(2)+2(-1) & 1(-3)+2(\frac{3}{2})\end{pmatrix}\end{array}$$

$$=\begin{pmatrix}0 & 0\\ 0 & 0\end{pmatrix}$$

Note: the integrality property does not hold in matrices!!! i.e., the following property does not hold

$$AB = 0 \to A = 0 \text{ or } B = 0$$

Exercise 8

(1) let

$$B = \begin{pmatrix} 1 & 5 & 2 \\ 0 & -2 & -4 \\ 3 & 0 & 5 \end{pmatrix}, C = \begin{pmatrix} 2 & 5 \\ 6 & 3 \\ 0 & -4 \end{pmatrix}$$

Find BC and $C^T B$.

Exercise 9

(2) let

$$A = \begin{pmatrix} 2 & 5 \\ -1 & 4 \\ 0 & -3 \end{pmatrix}, B = \begin{pmatrix} 6 & 10 & 7 \\ -2 & 3 & 1 \end{pmatrix}, C = \begin{pmatrix} 8 & -4 \\ -1 & 2 \end{pmatrix}$$

Find C^2 and $BA - 5I$.

Matrix representation of a system of linear equations

Example 10

Represent each of the following systems as product of matrices:

$$3x + y = 6$$
$$2x - 9y = 5$$

Solution:

$$\begin{pmatrix} 3 & 1 \\ 2 & -9 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 6 \\ 5 \end{pmatrix}$$

This is called the matrix form of a linear system.

Matrix representation of a system of linear equations

Example 11

Represent each of the following systems as product of matrices:

$$3x + y + z = 2$$
$$x - y + z = 4$$
$$5x - y + 2z = 12$$

Solution:

$$\begin{pmatrix} 3 & 1 & 1 \\ 1 & -1 & 1 \\ 5 & -1 & 2 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 2 \\ 4 \\ 12 \end{pmatrix}$$

Matrix Multiplication properties

Most of the usual properties of the multiplication on the real numbers hold

1. $\mathbf{0}A = \mathbf{0} = A\mathbf{0}$
2. $I_m A = A$ and $AI_n = A$
3. $AB = \mathbf{0}$, then not necessarily that $A = \mathbf{0}$ or $B = \mathbf{0}$
4. $AB \neq BA$
5. If $AB = AC$, then not necessarily $B = C$!
6. $(AB)^T = B^T A^T$