Section 2.3
Determinant of a matrix

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MATHS 211: Linear Algebra

Goal:

1. To define the determinant of a matrix.
2. To find the determinant of a matrix using cofactor expansion (Section 2.1).
3. To find the determinant of a matrix using row reduction (Section 2.2).
4. Explore the properties of the determinant and its relation to the inverse. (Section 2.3)
5. To solve linear system using the Cramer's rule. (Section 2.3)
6. The equation $A\mathbf{x} = \mathbf{b}$ (Section 2.3)

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Properties of the determinant

1. $$\det(kA) = k^n \det(A)$$

2. $$\det(A + B) \neq \det(A) + \det(B)$$

3. $$\det(AB) = \det(A) \cdot \det(B)$$

4. (Corollary)

$$\det(A^n) = (\det(A))^n, \qquad \det(A^{-1}) = \frac{1}{\det(A)}$$

5. If $\det(A) \neq 0$, then A has an inverse (invertible)

Example 1

Assume A is 5×5 matrix for which $\det(A) = -3$ Find the following:

1. $\det(3A)$

2. $\det(A^{-1})$

3. $\det(A^T)$

4. $\det(A^6)$

5. $\det((2A)^{-1})$

Solution:

Example 2

Use determinant to decide whether the given matrix is invertible or not

$$A = \begin{pmatrix} 2 & 0 & -1 \\ 0 & 2 & 3 \\ -1 & 0 & 5 \end{pmatrix}$$

Solution:

Example 3

Find the value(s) of k for which A is invertible.

$$A = \begin{pmatrix} 3 & k \\ k & 3 \end{pmatrix}, \qquad A = \begin{pmatrix} 2 & 1 & 0 \\ k & 2 & k \\ 2 & 4 & 2 \end{pmatrix}$$

Solution:

Adjoint matrix

Definition 4

Let $A \in \mathrm{Mat}(n, n, \mathbb{R})$ and C_{ij} is the cofacotr of a_{ij}, then the matrix with entries (C_{ij}) is called the **matrix of cofactors from** A. The transpose of this matrix is called the **adjoint** of A and is denoted by $adj(A)$.

Example 5

Use the adjoint method to find the inverse (if exists) to the following matrices:

$$A = \begin{pmatrix} -2 & 4 & 3 \\ 1 & 2 & 0 \\ 2 & -1 & -2 \end{pmatrix}$$

Theorem 6

$$A^{-1} = \frac{1}{\det(A)} adj(A)$$

Example 7

Use the adjoint method to find the inverse (if exists) to the following matrices:

$$A = \begin{pmatrix} -2 & 4 & 3 \\ 1 & 2 & 0 \\ 2 & -1 & -2 \end{pmatrix}$$

Solution:

Example 8

Use the adjoint method to find the inverse (if exists) to the following matrices:

$$A = \begin{pmatrix} 3 & 0 & 0 \\ -2 & 1 & 0 \\ 4 & 3 & 2 \end{pmatrix}$$

Solution:

Example 9

Use the adjoint method to find the inverse (if exists) to the following matrices:

$$A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{pmatrix}$$

Solution:

Cramer's Rule

Theorem 10

If $A\mathbf{x} = \mathbf{b}$ *is a system of* n *linear equations in* n *unknowns such that* $\det(A) =\neq 0$*, then the system has a unique solution given by*

$$x_1 = \frac{\det(A_1)}{\det(A)}, x_2 = \frac{\det(A_2)}{\det(A)}, \ldots x_n = \frac{\det(A_n)}{\det(A)},$$

where A_j *is the matrix obtained by replacing the entries in the* j*th column of* A *by the entries in the matrix*

$$\mathbf{b} = \begin{pmatrix} b_1 \\ b_2 \\ . \\ . \\ . \\ b_n \end{pmatrix}$$

Example 11

Solve using Cramer's rule the following system of linear equations

$$3x_1 + x_2 = 2$$
$$4x_1 + x_2 = 3$$

Solution:

Example 12

Solve using Cramer's rule the following system of linear equations

$$\begin{aligned} 3x_1 + 5x_2 &= 7 \\ 6x_1 + 2x_2 + 4x_3 &= 10 \\ -x_1 + 4x_2 - 3x_3 &= 0 \end{aligned}$$

Solution:

The equation $A\mathbf{x} = \mathbf{b}$

Theorem 13

The following are equivalent:

1. *A is invertible.*
2. $\det(A) \neq 0$.
3. *The reduced row echelon form is I_n.*
4. *$A\mathbf{x} = \mathbf{b}$ is consistent for every n matrix $\mathbf{b}$.*
5. *$A\mathbf{x} = \mathbf{b}$ has a unique solution for every n matrix $\mathbf{b}$.*