Section 4.3
Linear Independent Vectors

Dr. Abdulla Eid

College of Science

MATHS 211: Linear Algebra

Goal:

1. Define Linearly independent and linearly dependent.
2. From dependent to independent.
3.
4.

Subspace

Definition 1

Let V be a vector space. $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$ are called **linearly independent vectors** if the equation

$$k_1\mathbf{v}_1 + k_2\mathbf{v}_2 + \ldots k_n\mathbf{v}_n = \mathbf{0}$$

has only the unique solution $k_1 = 0, k_2 = 0, \ldots, k_n = 0$ (called the **trivial solution**).

Note: This means $k_1, k_2, \ldots, k_n$ are forced to be zero.

Definition 2

Let V be a vector space. $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$ are called **linearly dependent vectors** if the equation

$$k_1\mathbf{v}_1 + k_2\mathbf{v}_2 + \ldots k_n\mathbf{v}_n = \mathbf{0}$$

has other solution than $k_1 = 0, k_2 = 0, \ldots, k_n = 0$ (called the **nontrivial solution**).

Example 3

Determine whether the vectors $e_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$, $e_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$, $e_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$ are linearly independent in $\mathbb{R}^3$ or not.

Example 4

Determine whether the vectors $\mathbf{v}_1 = \begin{pmatrix} 1 \\ -2 \\ 3 \end{pmatrix}$, $\mathbf{v}_2 = \begin{pmatrix} 5 \\ 6 \\ -1 \end{pmatrix}$, $\mathbf{v}_3 = \begin{pmatrix} 3 \\ 2 \\ 1 \end{pmatrix}$ are linearly independent in $\mathbb{R}^3$ or not.

Example 5

Determine whether the vectors $\mathbf{v}_1 = \begin{pmatrix} 1 \\ 2 \\ 2 \\ -1 \end{pmatrix}$, $\mathbf{v}_2 = \begin{pmatrix} 4 \\ 9 \\ 9 \\ -4 \end{pmatrix}$, $\mathbf{v}_3 = \begin{pmatrix} 5 \\ 8 \\ 9 \\ -5 \end{pmatrix}$ are linearly independent in $\mathbb{R}^4$ or not.

Example 6

Determine whether the vectors $P_1 = 1$, $P_2 = X$, $P_3 = X^2$, $\ldots$, $P_n = X^n$ are linearly independent in $\mathbb{P}_n$ or not.

Example 7

Determine whether the vectors $P_1 = 1 - X$, $P_2 = 5 + 3X - 2X^2$, $P_3 = 1 + 3X - X^2$ are linearly independent in $\mathbb{P}_2$ or not.

Theorem 8

The set $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ *is linearly independent if and only if at least one of the vector is expressible as linear combination of the rest.*

Corollary 9

Let $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ *be a linearly dependent set with* $\mathbf{v}_1 = k_2\mathbf{v}_2 + \cdots + k_n\mathbf{v}_n$*, then*

$$span\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\} = span\{\mathbf{v}_2, \mathbf{v}_3, \ldots, \mathbf{v}_n\}$$

Example 10

Determine whether the vectors $\mathbf{v}_1 = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$, $\mathbf{v}_2 = \begin{pmatrix} 2 \\ 4 \\ 6 \end{pmatrix}$, $\mathbf{v}_3 = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$, $\mathbf{v}_4 = \begin{pmatrix} 1 \\ 4 \\ 5 \end{pmatrix}$ are linearly independent in $\mathbb{R}^3$ or not. If not, find an independent set from these vectors that gives the same span.

Theorem 11

The set $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_r\}$ *of vectors in* $\mathbb{R}^n$ *with* $r > n$ *is linearly dependent.*

Theorem 12

1. *A set containing* $\mathbf{0}$ *is linearly dependent.*
2. *A set with exactly one vector is linearly independent if and only if that vector is not* $\mathbf{0}$.
3. *A set with exactly two vectors if and only if neither vector is a scalar multiple of the other.*

Maps($\mathbb{R}, \mathbb{R}$)

Definition 13

Let $f_1, f_2, \ldots, f_n$ are functions that are $(n-1)$ differentiable functions. The determinant

$$W_{f_1,f_2,\ldots,f_n}(x) := \det \begin{pmatrix} f_1(x) & f_2(x) & \ldots & f_n(x) \\ f_1'(x) & f_2'(x) & \ldots & f_n'(x) \\ f_1''(x) & f_2''(x) & \ldots & f_n''(x) \\ . & . & \ldots & . \\ . & . & \ldots & . \\ . & . & \ldots & . \\ f_1^{n-1}(x) & f_2^{n-1}(x) & \ldots & f_n^{n-1}(x) \end{pmatrix}$$

is called the **Wronskian** of $f_1, f_2, \ldots, f_n$.

Theorem 14

If $f_1, f_2, \ldots, f_n$ have $n-1$ continuous derivatives with a **nonzero** *Wronskian, then these functions are linearly independent.*

Example 15

Determine whether the vectors $f_1 = 6$, $f_2 = 4\sin^2 x$, $f_3 = 3\cos^2 x$ are linearly independent in $\text{Maps}(\mathbb{R}, \mathbb{R})$ or not.

Example 16

Determine whether the vectors $f_1 = x$, $f_2 = e^x$, $f_3 = e^{-x}$ are linearly independent in $\text{Maps}(\mathbb{R}, \mathbb{R})$ or not.