Section 6.1
Inner Product

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MATHS 211: Linear Algebra

Goal:

1. Cauchy Schwarz Inequality.
2. Angle between vectors.
3. Properties of length and distance.
4. Orthogonality.
5. Orthogonal Complement.

Cauchy Schwarz Ineqality

Theorem 1

If $\mathbf{u}, \mathbf{v}$ *are two vectors, then*

$$|\langle \mathbf{u}, \mathbf{v} \rangle| \leq ||\mathbf{u}|| ||\mathbf{v}||$$

Proof: Let $a = ||\mathbf{u}||^2$, $b = 2\langle \mathbf{u}, \mathbf{v} \rangle$, and $c = \langle \mathbf{v}, \mathbf{v} \rangle$. Consider $\langle t\mathbf{u} + \mathbf{v}, t\mathbf{u} + \mathbf{v} \rangle \geq 0$ so the discriminant is less than or equal to zero.

Consequences of Cauchy Schwarz Inequality

$$|\langle \mathbf{u}, \mathbf{v}\rangle| \leq ||\mathbf{u}||\,||\mathbf{v}||$$

$$\frac{|\langle \mathbf{u}, \mathbf{v}\rangle|}{||\mathbf{u}||\,||\mathbf{v}||} \leq 1$$

$$-1 \leq \frac{\langle \mathbf{u}, \mathbf{v}\rangle}{||\mathbf{u}||\,||\mathbf{v}||} \leq 1$$

$$\cos\theta = \frac{\langle \mathbf{u}, \mathbf{v}\rangle}{||\mathbf{u}||\,||\mathbf{v}||} \text{ and } 0 \leq \theta \leq \pi$$

Definition 2

The angle θ between $\mathbf{u}$ and $\mathbf{v}$ is defined by

$$\theta = \cos^{-1}\left(\frac{\langle \mathbf{u}, \mathbf{v}\rangle}{||\mathbf{u}||\,||\mathbf{v}||}\right)$$

Find the angle

Example 3

Find the cosine of the angle between **u** and **v**, with the standard inner product.

1. $\mathbf{u} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$ and $\mathbf{v} = \begin{pmatrix} 4 \\ 4 \\ -4 \end{pmatrix}$.
2. $\mathbf{u} = \begin{pmatrix} 0 \\ 1 \\ 2 \\ 3 \end{pmatrix}$ and $\mathbf{v} = \begin{pmatrix} 5 \\ 2 \\ -3 \\ -4 \end{pmatrix}$.
3. $p = 1 + 3X - 5X^2$, $q = 2 - 3X^2$.
4. $U = \begin{pmatrix} 2 & 4 \\ 3 & 2 \end{pmatrix}, V = \begin{pmatrix} 0 & 1 \\ 0 & -2 \end{pmatrix}$

Properties of Length and Distance

Theorem 4

If $\mathbf{u}, \mathbf{v}, \mathbf{w}$ *are three vectors, then*

$$||\mathbf{u}+\mathbf{v}|| \leq ||\mathbf{u}|| + ||\mathbf{v}|| \quad (\textit{Triangle inequality for vectors})$$

$$d(\mathbf{u},\mathbf{v}) \leq d(\mathbf{u},\mathbf{w}) + d(\mathbf{v},\mathbf{w}) \quad (\textit{Triangle inequality for distances})$$

Proof: Consider $||\mathbf{u}+\mathbf{v}||^2$.

Orthogonality

Definition 5

Two vectors $\mathbf{u}, \mathbf{v}$ are called **orthogonal** if $\langle \mathbf{u}, \mathbf{v} \rangle = 0$.

Question: What is the angle between any two orthogonal vectors?

Check for orthongonality

Example 6

Determine whether **u** and **v** are orthogonal or not, with the standard inner product.

1. $\mathbf{u} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$ and $\mathbf{v} = \begin{pmatrix} 4 \\ 4 \\ -4 \end{pmatrix}$.
2. $\mathbf{u} = \begin{pmatrix} 0 \\ 1 \\ 2 \\ 3 \end{pmatrix}$ and $\mathbf{v} = \begin{pmatrix} 5 \\ 2 \\ -3 \\ -4 \end{pmatrix}$.
3. $f = x,\ q = X^2$ on $[-1, 1]$.
4. $U = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}, V = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$

Pythagorean Theorem

Theorem 7

If $\mathbf{u}, \mathbf{v}$ *are orthogonal vectors, then*

$$||\mathbf{u} + \mathbf{v}||^2 = ||\mathbf{u}|| + ||\mathbf{v}||$$

Proof: Consider $||\mathbf{u} + \mathbf{v}||^2 = \langle \mathbf{u} + \mathbf{v}, \mathbf{u} + \mathbf{v} \rangle$.

Example 8

Show that if $\mathbf{u}, \mathbf{v}$ are orthogonal unit vectors in V, then $||\mathbf{u} - \mathbf{v}|| = \sqrt{2}$.

Orthogonal Complement

Definition 9

If W is a subspace of an inner vector space V, then the set of all vectors in V that are orthogonal to every vector in W is called the **orthogonal complement** of W and is denoted by $W^\perp$.

$$W^\perp := \{\hat{w} \in V \mid \langle \hat{w}, w \rangle = 0, \text{ for all } w \in W\}$$

Theorem 10

1. *$W^\perp$ is a subspace of W.*
2. $W \cap W^\perp = \{\mathbf{0}\}$
3. $(W^\perp)^\perp$

Row space and null space are orthogonal

Example 11

Let $W = \text{span}\{\mathbf{w}_1, \mathbf{w}_2, \mathbf{w}_3\}$ where,

$$\mathbf{w}_1 = \begin{pmatrix} 2 \\ 1 \\ 3 \end{pmatrix}, \mathbf{w}_2 = \begin{pmatrix} -1 \\ -4 \\ 2 \end{pmatrix}, \mathbf{w}_3 = \begin{pmatrix} 4 \\ -5 \\ 13 \end{pmatrix}$$

Row space and null space are orthogonal

Example 12

Let $W = \text{span}\{\mathbf{w}_1, \mathbf{w}_2, \mathbf{w}_3\}$ where,

$$\mathbf{w}_1 = \begin{pmatrix} 3 \\ 0 \\ 1 \\ -2 \end{pmatrix}, \mathbf{w}_2 = \begin{pmatrix} -1 \\ -2 \\ -2 \\ 1 \end{pmatrix}, \mathbf{w}_3 = \begin{pmatrix} 4 \\ 2 \\ 3 \\ -3 \end{pmatrix}$$