Section 8.1
Linear Transformation

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MATHS 211: Linear Algebra

Goal:

1. Define Linear Transformations.
2. Examples of Linear Transformations.
3. Finding the linear transformation from a basis.
4. Kernel and Range of a linear transformations.
5. Properties and of Kernel and Images.
6. Rank and Nullity of a linear transformation.

Definition 1

A **linear transformation** is a function $T : V \to W$ with

1. $T(k\mathbf{v}) = kT(\mathbf{v})$. (Homogeneity Property)
2. $T(\mathbf{u}+\mathbf{v}) = T(\mathbf{u}) + T(\mathbf{v})$. (Additive Property)

In the special case, where $V = W$, the linear transformation T is called **linear operator**.

Simple Consequences from the definition

- $$T(k_1\mathbf{v}_1 + k_2\mathbf{v}_2) = T(k_1\mathbf{v}_1) + T(k_2\mathbf{v}_2)$$
  $$= k_1 T(\mathbf{v}_1) + k_2 T(\mathbf{v}_2)$$
- In general,
  $$T(k_1\mathbf{v}_1 + k_2\mathbf{v}_2 + \cdots + k_n\mathbf{v}_n) = k_1 T(\mathbf{v}_1) + k_2 T(\mathbf{v}_2) + \cdots + k_n T(\mathbf{v}_n)$$
- $$T(\mathbf{0}) = T(\mathbf{0} + \mathbf{0})$$
  $$T(\mathbf{0}) = T(\mathbf{0}) + T(\mathbf{0})$$
  $$\mathbf{0}_W = T(\mathbf{0}) \qquad \text{Zero goes to zero}$$
- $$T(\mathbf{u} - \mathbf{v}) = T(\mathbf{u}) - T(\mathbf{v})$$

Matrix Transformation

Example 2

Let A be any $m \times n$ matrix. Define the linear transformation $T_A : \mathbb{R}^n \to \mathbb{R}^m$ by

$$T_A(\mathbf{x}) = A\mathbf{x}$$

Zero Transformation

Example 3

Let V be any vector space. Define the linear transformation $T : V \to W$ by

$$T(\mathbf{x}) = \mathbf{0}_W$$

Identity Operator

Example 4

Let V be any vector space. Define the linear transformation (operator) $T : V \to V$ by

$$T(\mathbf{x}) = \mathbf{x}$$

Dilation and Contraction Operators

Example 5

Let V be any vector space. Define the linear operator $T_k : V \to V$ by

$$T_k(\mathbf{x}) = k\mathbf{x}$$

If $0 < k < 1$, then T_k is called the **contraction** of V with factor k and if $k > 1$, it is called the **dilation** of V with factor k.

A Linear Transformation from $\mathbb{P}_n$ to $\mathbb{P}_{n+1}$

Example 6

Define the linear operator $T : \mathbb{P}_n \to \mathbb{P}_n$ by

$$T(p(X)) = XP(X)$$

$$T(c_0 + c_1X + c_2X^2 + \cdots + c_nX^n) = c_0X + c_1X^2 + c_2X^3 + \cdots + c_nX^{n+1}$$

Linear Transformation of the Matrices

Example 7

Define the linear transformation $T : \text{Mat}(m, n, \mathbb{R}) \to \text{Mat}(n, m, \mathbb{R})$ by

$$T(A) = A^T$$

Linear Transformation of the Matrices

Example 8

Define the function $T : \mathrm{Mat}(m, n, \mathbb{R}) \to \mathbb{R}$ by

$$T(A) = \det(A)$$

Is T a linear transformation?

Evaluation Transformation

Example 9

Let $c_1, c_2, \ldots, c_n$ be real numbers. Define the linear transformation $T : \text{Maps}(\mathbb{R}, \mathbb{R}) \to \mathbb{R}^n$ by

$$T(f) = (f(c_1), f(c_2), \ldots, f(c_n))$$

Finding Linear Transformation from the image of a basis

Example 10

Consider the basis $\mathcal{B} = \{\mathbf{v}_1, \mathbf{v}_2\}$ of $\mathbb{R}^2$, where

$$\mathbf{v}_1 = \begin{pmatrix} 2 \\ 1 \end{pmatrix}, \quad \mathbf{v}_2 = \begin{pmatrix} -1 \\ 3 \end{pmatrix}$$

and let $T : \mathbb{R}^2 \to \mathbb{R}^3$ be the linear transformation such that

$$T(\mathbf{v}_1) = \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix}, \quad T(\mathbf{v}_2) = \begin{pmatrix} 0 \\ 3 \\ 5 \end{pmatrix}$$

Find a formula for $T(x_1, x_2)$ in general and use it to find $T(2, -3)$ and $T(4, -1)$.

Finding Linear Transformation from the image of a basis

Example 11

Consider the basis $\mathcal{B} = \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ of $\mathbb{R}^3$, where

$$\mathbf{v}_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \quad \mathbf{v}_2 = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \quad \mathbf{v}_3 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

and let $T : \mathbb{R}^3 \to \mathbb{R}^3$ be the linear transformation such that

$$T(\mathbf{v}_1) = \begin{pmatrix} -1 \\ 2 \\ 4 \end{pmatrix}, \quad T(\mathbf{v}_2) = \begin{pmatrix} 0 \\ 3 \\ 2 \end{pmatrix}, \quad T(\mathbf{v}_3) = \begin{pmatrix} 1 \\ 5 \\ -1 \end{pmatrix}.$$

Find a formula for $T(x_1, x_2, x_3)$ in general and use it to find $T(2, 4, -1)$ and $T(4, 3, 4)$.

Definition 12

If $T : V \to W$ be a linear transformation. The set of all vectors in V that T maps into zero is called the **kernel** of T and denoted by $ker(T)$.the set of all vectors in W that are images under T of at least one vector in V is called the **range** of T and is denoted by $R(T)$.

$$ker(T) := \{\mathbf{v} \in V \mid T(\mathbf{v}) = \mathbf{0}_W\}$$

$$R(T) := \{\mathbf{w} \in W \mid T(\mathbf{v}) = \mathbf{w}, \text{ for some } \mathbf{v} \in V\}$$

Matrix Transformation

Example 13

Let A be any $m \times n$ matrix. Define the linear transformation $T_A : \mathbb{R}^n \to \mathbb{R}^m$ by

$$T_A(\mathbf{x}) = A\mathbf{x}$$

$Ker(T_A)$ is the Null space of A while $R(T_A)$ is the Column space of A

Zero Transformation

Example 14

Let V be any vector space. Define the linear transformation $T : V \to W$ by

$$T(\mathbf{x}) = \mathbf{0}_W$$

Identity Operator

Example 15

Let V be any vector space. Define the linear transformation (operator) $T : V \rightarrow V$ by

$$T(\mathbf{x}) = \mathbf{x}$$

Dilation and Contraction Operators

Example 16

Let V be any vector space. Define the linear operator $T_k : V \to V$ by

$$T_k(\mathbf{x}) = k\mathbf{x}$$

If $0 < k < 1$, then T_k is called the **contraction** of V with factor k and if $k > 1$, it is called the **dilation** of V with factor k.

A Linear Transformation from $\mathbb{P}_n$ to $\mathbb{P}_{n+1}$

Example 17

Define the linear operator $T : \mathbb{P}_n \to \mathbb{P}_n$ by

$$T(p(X)) = XP(X)$$

$$T(c_0 + c_1X + c_2X^2 + \cdots + c_nX^n) = c_0X + c_1X^2 + c_2X^3 + \cdots + c_nX^{n+1}$$

Theorem 18

Let $T : V \to W$ be a linear transformation. $Ker(T)$ and $R(A)$ are both subspaces of V and W.

1. Dimension of the $ker(T)$ is called the **nullity** of T.
2. Dimension of the $R(A)$ is called the **rank** of T.

Theorem 19

Dimension Theorem Let $T : V \to W$ be a linear transformation, then

$$rank(T) + Nullity(T) = \dim(V) = n$$

Example 20

Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be the linear transformation given by

$$T(x, y) = (x - 3y, -2x + 6y)$$

Which of the following vectors are in $R(T)$?
$(1, -2)$, $(3, 1)$, $(-2, 4)$ Which of the following vectors are in $ker(T)$?
$(1, -3)$, $(3, 1)$, $(-6, -2)$ Find a basis for the $ker(T)$? Find a basis for the $R(T)$? Verify the formula in the dimension theorem?

Example 21

Let $T : \mathbb{P}_2 \to \mathbb{P}_3$ be the linear transformation given by

$$T(p(X)) = (X + 1)p(X)$$

Which of the following vectors are in $R(T)$?
$X + X^2$, $1 + X$, $3 - X^2$ Which of the following vectors are in $ker(T)$?
X^2, 0, $X + 1$ Find a basis for the $ker(T)$? Find a basis for the $R(T)$?
Verify the formula in the dimension theorem?