Section 2.4
One sided limits
2 Lectures

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MATHS 101: Calculus I

Goals:

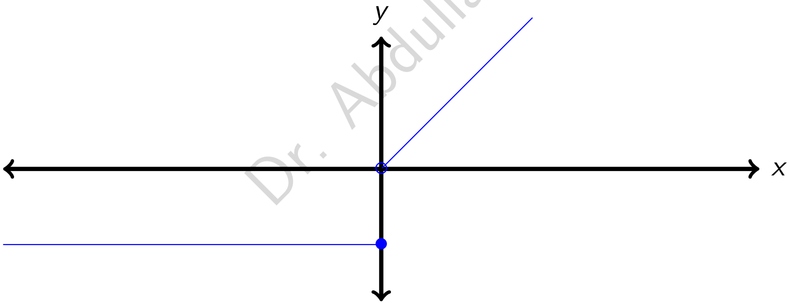
1. Finding the one–sided limit algebraically.
2. Finding the one–sided limit graphically.

Motivational Example

Consider the function

$$f(x) = \begin{cases} x, & x > 0 \\ -1, & x \leq 0 \end{cases}$$

The graph of the function is given by



Question: What can you say about the $\lim_{x \to 0} f(x)$?

Continue...

1. If we approaches 0 from the **left** (values slightly less than 0), $f(x)$ approaches -1. In this case, we say that the **limit from the left** is equal to -1 and we write it as

$$\lim_{x \to 0^-} f(x) = -1$$

2. If we approaches 0 from the **right** (values slightly greater than 0), $f(x)$ approaches 0. In this case, we say that the **limit from the left** is equal to 0 and we write it as

$$\lim_{x \to 0^+} f(x) = 0$$

Theorem 1

$$\lim_{x \to a} f(x) = L \iff \lim_{x \to a^+} f(x) = L \quad \textit{and} \quad \lim_{x \to a^-} f(x) = L$$

Otherwise, we say $\lim_{x \to a} f(x)$ Does Not Exist.

1 - Finding the one–sided limit algebraically

Example 2

Consider

$$f(x) = \begin{cases} \frac{x^2-4}{x-2}, & x < 2 \\ 3x+1, & x \geq 2 \end{cases}$$

Solution:

1. $\lim_{x\to 2^+} f(x) = \lim_{x\to 2^+} 3x+1 = 7.$
2. $\lim_{x\to 2^-} f(x) = \lim_{x\to 2^-} \frac{x^2-4}{x-2} = \lim_{x\to 2^-} \frac{(x-2)(x+2)}{(x-2)} =$ $\lim_{x\to 2^-}(x+2) = 4.$
3. $\lim_{x\to 2} f(x) =$ Does Not Exist, since $\lim_{x\to 2^-} f(x) \neq \lim_{x\to 2^+} f(x).$
4. $\lim_{x\to -4} f(x) = \lim_{x\to -4} \frac{x^2-4}{x-2} = = -6.$
5. $\lim_{x\to 0^+} f(x) = \lim_{x\to 0^+} 3x+1 = 1.$

Example 3

Consider

$$f(x) = \begin{cases} \frac{x^2-3x+2}{x-1}, & x \neq 1 \\ 5x^2, & x = 1 \end{cases}$$

Solution:

1. $\lim_{x\to 1} f(x)$. Since the function has more than one definition near $x = 1$, we need to find the left and the right limits.
   1. $\lim_{x\to 1^-} f(x) = \lim_{x\to 1^-} \frac{x^2-3x+2}{x-1} = = \lim_{x\to 1^-} \frac{(x-1)(x+2)}{(x-1)} = \lim_{x\to 1^-}(x+2) = 3.$
   2. $\lim_{x\to 1^+} f(x) = \lim_{x\to 1^+} \frac{x^2-3x+2}{x-1} = = \lim_{x\to 1^+} \frac{(x-1)(x+2)}{(x-1)} = \lim_{x\to 1^+}(x+2) = 3.$

   Hence $\lim_{x\to 1} f(x) = 3$, since $\lim_{x\to 1^-} f(x) = \lim_{x\to 1^+} f(x)$.
2. $\lim_{x\to 4} f(x) = \lim_{x\to 4} \frac{x^2-3x+2}{x-1} = 2.$
3. $\lim_{x\to -1^+} f(x) = \lim_{x\to -1+} \frac{x^2-3x+2}{x-1} = -3.$

Exercise 4

Consider

$$f(x) = \begin{cases} \frac{x^2-4}{x-2}, & 1.5 < x < 2 \\ 5x^2+1, & x \geq 2 \\ \frac{x^2-1}{x-1}, & x \neq 1 \\ 3, \quad x < 0 \end{cases}$$

1. $\lim_{x\to 1^+} f(x)$
2. $\lim_{x\to 1^-} f(x)$
3. $\lim_{x\to 1} f(x)$
4. $\lim_{x\to 2} f(x)$
5. $\lim_{x\to 0} f(x)$
6. $\lim_{x\to 1.5} f(x)$
7. $\lim_{x\to -1} f(x)$

Example 5

Consider $f(x) = |x|$, which can be written as

$$f(x) = |x| = \begin{cases} x, & x \geq 0 \\ -x, & x < 0 \end{cases}$$

Solution:

1. $\lim_{x \to 0^+} f(x) = \lim_{x \to 0^+} x = 0.$
2. $\lim_{x \to 0^-} f(x) = \lim_{x \to 0^-} -x = 0.$
3. $\lim_{x \to 0} f(x) = 0$, since $\lim_{x \to 0^-} f(x) = \lim_{x \to 0^+} f(x)$.

Exercise 6

Find

$$\lim_{x\to -2^+} (x+3)\frac{|x+2|}{(x+2)}$$

Solution:

$$\lim_{x\to -2^+} (x+3)\frac{|x+2|}{(x+2)} = \lim_{x\to -2^+} (x+3)\frac{(x+2)}{(x+2)}$$

$$\lim_{x\to -2^+} (x+3) = 1$$

Example 7

Find $\lim_{x\to 0^-} \frac{\sqrt{x^2}}{x}$

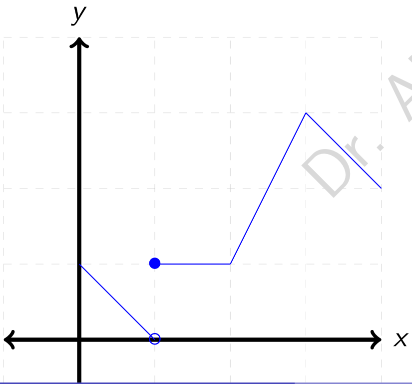
Solution:

$$\lim_{x\to 0^-} \frac{\sqrt{x^2}}{x} = \lim_{x\to 0^-} \frac{|x|}{x} \lim_{x\to 0^-} \frac{-x}{x} = -1$$

2 - Finding the one–sided limit graphically

Example 8

Consider the function



Find

1. $\lim_{x\to 0} f(x)$
2. $\lim_{x\to 1} f(x)$
3. $\lim_{x\to 2} f(x)$
4. $\lim_{x\to 3} f(x)$
5. $\lim_{x\to 4} f(x)$