Section 3.2
Definition of the Derivative
2 Lectures

Dr. Abdulla Eid

College of Science

MATHS 101: Calculus I

Nondifferentiable function with a corner

Example 1

Show that the function

$$f(x) = |x| = \begin{cases} x, & x \geq 0 \\ -x, & x < 0 \end{cases}$$

is not differentiable at $x = 0$.

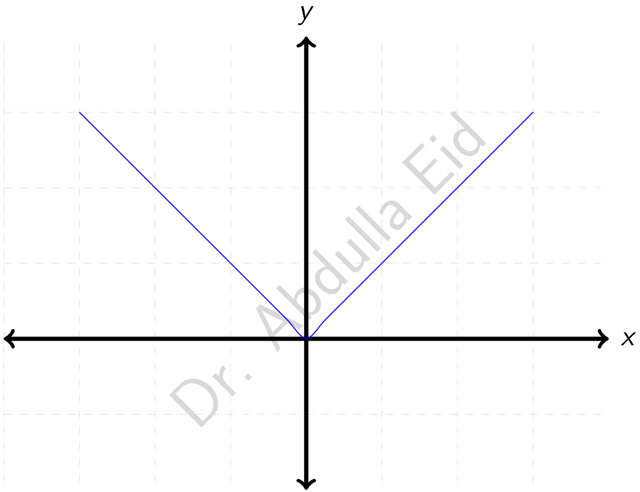
Solution: We will try to find the derivative of the function at $x = 0$,

$$f'(0) = \lim_{h \to 0} \frac{f(0+h) - f(0)}{h}$$

$$f'(0) = \lim_{h \to 0} \frac{|0+h| - |0|}{h}$$

$$f'(0) = \lim_{h \to 0} \frac{|h|}{h} \text{ which is does not exist by Example } \textbf{??}$$

So the function $f(x) = |x|$ is not differentiable at $x = 0$ (since we have a corner ! at $x = 0$).


y
x

Nondifferentiable function with a cusp

Example 2

Show that the function $f(x) = \sqrt{|x|}$ is not differentiable at $x = 0$.

Solution: We will try to find the derivative of the function at $x = 0$,

$$f'(0) = \lim_{h \to 0} \frac{f(0+h) - f(0)}{h}$$

$$f'(0) = \lim_{h \to 0} \frac{\sqrt{|h|} - 0}{h}$$

We need to find the left and the right limit.

$$f'(0) = \lim_{h \to 0} \frac{\sqrt{|h|} - 0}{h}$$

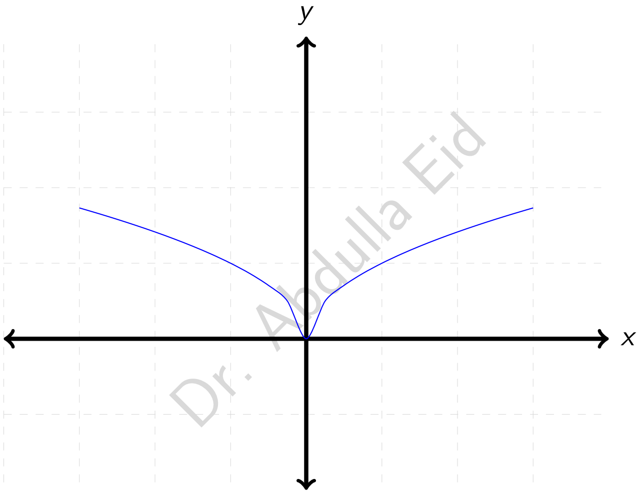
$$\begin{aligned} \lim_{h \to 0^+} \frac{\sqrt{|h|} - 0}{h} &= \lim_{h \to 0^+} \frac{\sqrt{h} - 0}{h} \\ &= \lim_{h \to 0^+} \frac{1}{\sqrt{h}} \\ &= \infty \end{aligned}$$

$$\begin{aligned} \lim_{h \to 0^-} \frac{\sqrt{|h|} - 0}{h} &= \lim_{h \to 0^-} \frac{\sqrt{-h} - 0}{h} \\ &= \lim_{h \to 0^-} \frac{1}{\sqrt{-h}} \\ &= \infty \end{aligned}$$

so we have

$$f'(0) = \infty$$

i.e., the slope is infinity so the function $f(x) = \sqrt{|x|}$ is not differentiable at $x = 0$ (since we have a cusp ! at $x = 0$).


y
x

Nondifferentiable everywhere function

Example 3

Look at the function that results from everyday trading in stock, or FX, you will see that at every point we have either a cusp or corner. Example, look at `www.finance.yahoo.com`

Differentiablity $\rightarrow$ Continuity

Theorem 4

If f is differentiable at $x = a$, then f is continuous at $x = a$.

Solution: Assume f is differentiable at $x = a$, i.e., the derivative $f'(a)$ (as a limit exist). We want to show that f is continuous at $x = a$,i.e., $\lim_{x\to a} f(x) = f(a)$.

$$f(x) - f(a) = \frac{f(x) - f(a)}{(x - a)} \cdot (x - a)$$

$$\lim_{x\to a} f(x) - f(a) = \lim_{x\to a} \frac{f(x) - f(a)}{(x - a)} \cdot (x - a)$$

$$\lim_{x\to a} f(x) - f(a) = f'(a) \cdot 0 = 0$$

$$\lim_{x\to a} f(x) = f(a)$$

Therefore, the function is continuous at $x = a$.