# Section 3.3 Product and Quotient rules 1 Lecture

Dr. Abdulla Eid

College of Science

MATHS 101: Calculus I

### Motivation

Goal: We want to derive rules to find the derivative of product f(x)g(x) and quotient  $\frac{f(x)}{g(x)}$  of two functions.

#### Example 1

We want to find (in a general way) the derivative of the functions:

• 
$$f(x) = (3x+1)(5x+2).$$
  
•  $f(x) = xe^{x}.$   
•  $f(x) = \frac{3x+1}{x^3+2x+1}.$   
•  $f(x) = \frac{x+1}{e^{x}+6x-3}.$ 

## The Product Rule (Leibniz Rule)

#### Theorem 2

$$(f(x)g(x))' = f'(x)g(x) + f(x)g'(x).$$

(f(x)g(x))' = (derivative of first)(second) + (first)(derivative of second)

Before we prove this theorem, recall that the definition of the derivative is

\_ .( .

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$
 and  $g'(x) = \lim_{h \to 0} \frac{g(x+h) - g(x)}{h}$ 

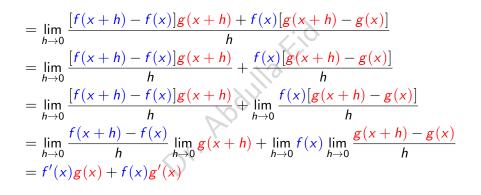
Proof: Let F(x) = f(x)g(x). Then,

$$F'(x) = \lim_{h \to 0} \frac{F(x+h) - F(x)}{h}$$
$$= \lim_{h \to 0} \frac{f(x+h)g(x+h) - f(x)g(x)}{h}$$

We will use a "trick" by adding and subtracting f(x)g(x+h) in the middle of the numerator.

$$F'(x) = \lim_{h \to 0} \frac{f(x+h)g(x+h) - f(x)g(x+h) + f(x)g(x+h) - f(x)g(x)}{h}$$
$$= \lim_{h \to 0} \frac{[f(x+h) - f(x)]g(x+h) + f(x)[g(x+h) - g(x)]}{h}$$

### Continue...



#### Example 3

Find the derivative of each of the following:

• 
$$F(x) = (x^2 + 5x - 6)(6x^2 - 5x + 6)$$
  
•  $F(x) = 2(\sqrt{x} + 5x - 3)(\sqrt[4]{x} - 4\sqrt{x})$ 

Solution: (1)

F'(x) = (derivative of first) (second) + (first) (derivative of second) $= (2x+5)(6x^2 - 5x + 6) + (x^2 + 5x - 6)(12x - 5)$ 

(2)

F'(x) = (derivative of first) (second) + (first) (derivative of second) $= 2 \left[ \left( \frac{1}{2\sqrt{x}} + 5 \right) \left( \sqrt[4]{x} - 4\sqrt{x} \right) + \left( \sqrt{x} + 5x - 3 \right) \left( \frac{1}{4}x^{\frac{-3}{4}} - \frac{4}{2\sqrt{x}} \right) \right]$ 

### Product rule for 3 functions or more

$$(fg)' = f'g + fg'$$

$$(fgh)' = f'gh + fg'h + fgh'$$

### Example 4

Find the derivative of each of the following:

• 
$$F(x) = (x-1)(x-2)(x^2-4)$$

Solution:

$$F'(x) = (1)(x-2)(x^2-4) + (x-1)(1)(x^2-4) + (x-1)(x-2)(2x)$$

# The Quotient Rule

#### Theorem 5

$$\left(\frac{f(x)}{g(x)}\right)' = \frac{g(x)f'(x) - f(x)g'(x)}{g(x)^2}$$

(denominator)(derivative of numerator) – (numerator)(derivative of denomin (denominator)<sup>2</sup>

To prove this theorem, we will use the product rule.

Proof: Let  $F(x) = \frac{f(x)}{g(x)}$ . We want to find F'(x). For that we apply the product rule to

$$F(x)g(x) = f(x)$$

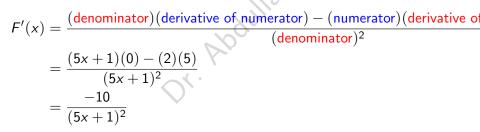
(derivative of first)(second) + (first)(derivative of second) = f'(x)F'(x)g(x) + F(x)g'(x) = f'(x)F'(x)g(x) = f'(x) - F(x)g'(x) $F'(x) = \frac{f'(x) - F(x)g'(x)}{g(x)}$  $F'(x) = \frac{f'(x) - \frac{f(x)}{g(x)}g'(x)}{g(x)}$  $F'(x) = \frac{g(x)f'(x) - f(x)g'(x)}{(g(x))^2}$ 

### Example 6

Find the derivative of each of the following:

**1** 
$$F(x) = \frac{2}{5x+1}$$
  
**2**  $F(x) = \frac{1-x}{1-x^3}$ 

Solution: (1)



### Continue...

Recall we want to find the derivative of  $F(x) = \frac{1-x}{1-x^3}$ .

 $F'(x) = \frac{(\text{denominator})(\text{derivative of numerator}) - (\text{numerator})(\text{derivative or})}{(\text{denominator})^2}$  $= \frac{(1-x^3)(-1) - (1-x)(-3x^2)}{(1-x^3)^2}$  $= \frac{-1+x^3+3x^2-3x^3}{(1-x^3)^2}$  $= \frac{-1+3x^2-2x^3}{(1-x^3)^2}$ 

### Exercise 7

Find the derivative of the following functions:

$$f(x) = \frac{e^x}{x}$$

$$f(x) = \frac{ax+b}{cx+d}$$

Or. Applulia Ein