Section 3.5
Derivative of Trigonometric Functions
2 Lectures

Dr. Abdulla Eid

College of Science

MATHS 101: Calculus I

1. Review of the trigonometric functions (Pre–Calculus).
2. Limits involving trigonometric functions.
3. Derivative of the basic trigonometric functions.
4. Derivative of the functions that involve trigonometric functions.

Review of the trigonometric functions (Pre–Calculus)

1. Radian and degree of an angle.
2. Definition of sine and cosine functions and their graphs.
3. Definition of the other trigonometric functions.
4. Some important trigonometric identities.

Radian and Degree

- Angles are usually measured by their degree, for example, 30°, 45°, 90°, 180°, etc.
- On the other hand, angles as real numbers are given in terms of radian.

degree $\to$ radian

$$\frac{\text{degree}}{180^\circ} \cdot \pi$$

radian $\to$ degree

$$\frac{\text{radian}}{\pi} \cdot 180^\circ$$

degree $\to$ radian

$$\frac{\text{degree}}{180^\circ} \cdot \pi$$

radian $\to$ degree

$$\frac{\text{radian}}{\pi} \cdot 180^\circ$$

Exercise 1

Fill in the following table

Degree	0°	30°		60°	90°	120°		180°	270°	360°
Radian			$\frac{\pi}{4}$				$\frac{3\pi}{4}$			

Definition of Sine and Cosine

Unit Circe

Right Triangle

$$\sin\theta = \frac{\text{opposite}}{\text{hypotunse}}$$

$$\cos\theta = \frac{\text{adjacent}}{\text{hypotunse}}$$

$$-1 \leq \sin\theta, \cos\theta \leq 1$$

Example 2

Compute using the unit circle the values of $\sin\frac{\pi}{2}$, $\cos\frac{\pi}{2}$, $\sin\pi$, $\cos\pi$.

Solution:

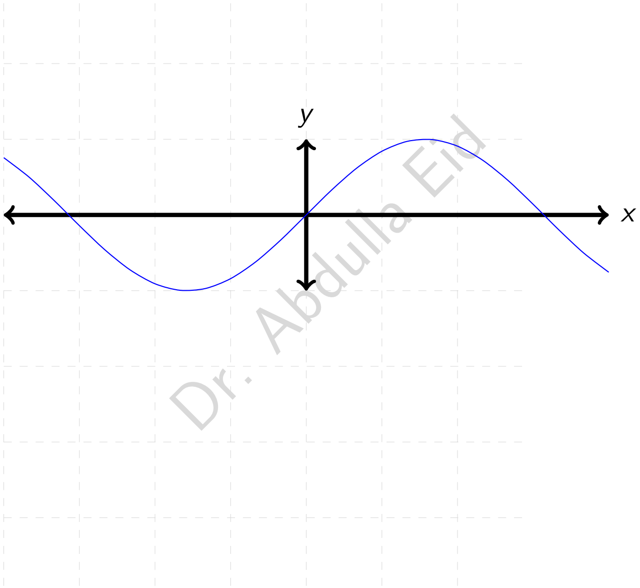
$$\begin{aligned}\sin\frac{\pi}{2} &= \\ \cos\frac{\pi}{2} &= \\ \sin\pi &= \\ \cos\pi &= \end{aligned}$$

Exercise 3

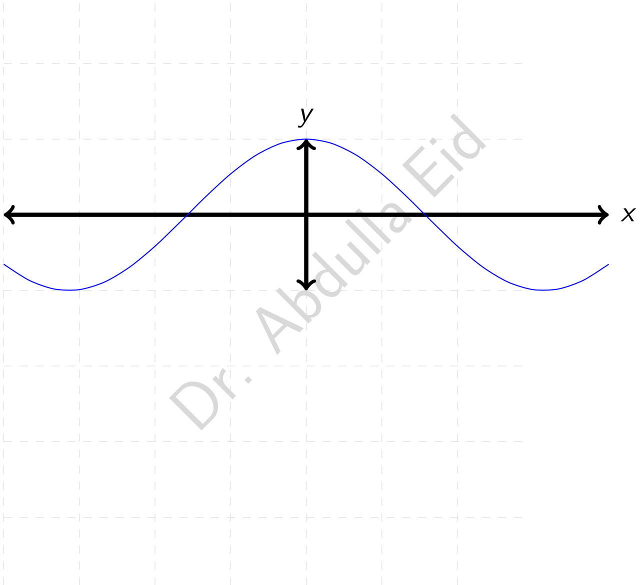
Fill in the following table using a calculator

θ	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$	$\frac{3\pi}{4}$	$\frac{5\pi}{6}$	π	$\frac{3\pi}{2}$	2π
$\sin\theta$											
$\cos\theta$											

Graph of since and cosine



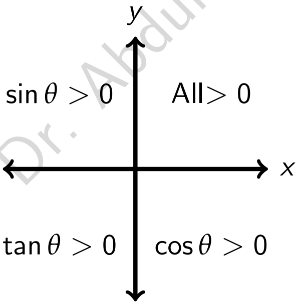
Graph of since and cosine



3 - Definition of the other trigonometric functions

$$\tan\theta = \frac{\sin\theta}{\cos\theta} = \frac{\text{opposite}}{\text{adjacent}} \qquad \cot\theta = \frac{\cos\theta}{\sin\theta} = \frac{\text{adjacent}}{\text{opposite}}$$

$$\sec\theta = \frac{1}{\cos\theta} = \frac{\text{hypotunse}}{\text{adjacent}} \qquad \csc\theta = \frac{1}{\sin\theta} = \frac{\text{hypotunse}}{\text{opposite}}$$



“**A**ll **S**tudents **T**ake **C**alculus“

Some useful trigonometric identities

1. $\sin^2\theta + \cos^2\theta = 1.$

2. $1 + \tan^2\theta = \sec^2\theta \qquad (1 + \cot^2\theta = \csc^2\theta).$

3. $\sin(-\theta) = -\sin\theta$ (odd function)
   $\cos(-\theta) = \cos\theta$ (even function).

4. Double angle formula:
   $\sin(2\theta) = 2\sin\theta\cos\theta$.
   $\cos(2\theta) = \cos^2(\theta) - \sin^2(\theta)$.
   $\cos^2\theta = \frac{1}{2}(1 + \cos(2\theta)).$
   $\sin^2\theta = \frac{1}{2}(1 - cos(2\theta)).$

5. $\sin(a + b) = \sin a\cos b + \cos a\sin b.$
   $\cos(a + b) = \cos a\cos b - \sin a\sin b.$
   These are very useful formula!

Continuity of Sine and Cosine

Exercise 4

Prove that f is continuous at a if and only if

$$\lim_{h\to 0} f(a+h) = f(a)$$

Exercise 5

Use the trigonometric identities to show that

$$\lim_{h\to 0} \sin(a+h) = \sin(a)$$

and use the exercise above to show that $f(x) = \sin x$ is a continuous function. Do the same for the $f(x) = \cos x$.

2 - Limits involving trigonometric functions

Example 6

$$\lim_{\theta \to 0} \frac{\sin \theta}{\theta} = 1$$

Solution:

$$\begin{aligned}
\text{area of } &\leq \text{area of } \leq \text{area of} \\
\frac{1}{2}\sin\theta &\leq \frac{1}{2}\theta \leq \frac{1}{2}\tan\theta \\
1 &\leq \frac{\theta}{\sin\theta} \leq \frac{\tan\theta}{\sin\theta} \\
1 \geq \frac{\sin\theta}{\theta} \geq \cos\theta \to &\lim_{\theta\to 0} \frac{\sin\theta}{\theta} = 1
\end{aligned}$$

Example 7

Find

$$\lim_{\theta\to 0}\frac{1-\cos\theta}{\theta}$$

Solution:

$$\begin{aligned}
\lim_{\theta\to 0}\frac{1-\cos\theta}{\theta} &= \lim_{\theta\to 0}\frac{1-\cos\theta}{\theta}\cdot\frac{1+\cos\theta}{1+\cos\theta}\\
&= \lim_{\theta\to 0}\frac{1-\cos^2\theta}{\theta(1+\cos\theta)}\\
&= \lim_{\theta\to 0}\frac{\sin^2\theta}{\theta(1+\cos\theta)}\\
&= \lim_{\theta\to 0}\frac{\sin\theta}{\theta}\cdot\frac{\sin\theta}{(1+\cos\theta)}\\
&= 1\cdot 0\\
&= 0
\end{aligned}$$

Exercise 8

Find

$$\lim_{\theta \to 0} \frac{\tan\theta}{\theta}$$

Solution:

$$\begin{aligned}
\lim_{\theta \to 0} \frac{\tan\theta}{\theta} &= \lim_{\theta \to 0} \frac{\sin\theta}{\theta\cos\theta} \\
&= \lim_{\theta \to 0} \frac{\sin\theta}{\theta} \cdot \frac{1}{(\cos\theta)} \\
&= 1 \cdot 1 \\
&= 1
\end{aligned}$$

Three Important limits

$$\lim_{\theta \to 0} \frac{\sin \theta}{\theta} = 1$$

$$\lim_{\theta \to 0} \frac{1 - \cos \theta}{\theta} = 0$$

$$\lim_{\theta \to 0} \frac{\tan \theta}{\theta} = 1$$

Example 9

Find

$$\lim_{x \to 0} \frac{\sin(7x)}{x}$$

Solution:

$$\begin{aligned}\lim_{x \to 0} \frac{\sin(7x)}{x} &= \lim_{x \to 0} \frac{7\sin(7x)}{7x} \\ &= 7 \lim_{x \to 0} \frac{\sin(7x)}{7x} \\ &= 7\end{aligned}$$

Example 10

Find

$$\lim_{x\to 0} x\sin\left(\frac{1}{x}\right)$$

Solution:

$$-1 \le \sin\left(\frac{1}{x}\right) \le 1$$

$$-x \le x\sin\left(\frac{1}{x}\right) \le x$$

$$\lim_{x\to 0} -x \le \lim_{x\to 0} x\sin\left(\frac{1}{x}\right) \le \lim_{x\to 0} x$$

$$0 \le \lim_{x\to 0} x\sin\left(\frac{1}{x}\right) \le 0$$

$$\lim_{x\to 0} x\sin\left(\frac{1}{x}\right) = 0$$

Exercise 11

Find

$$\lim_{x\to 0^+} \sqrt{x}e^{\sin\left(\frac{\pi}{x}\right)}$$

Solution:

$$\begin{aligned}
-1 &\leq \sin\left(\frac{\pi}{x}\right) \leq 1 \\
e^{-1} &\leq e^{\sin\left(\frac{\pi}{x}\right)} \leq e^{1} \\
-\sqrt{x}e^{-1} &\leq \sqrt{x}e^{\sin\left(\frac{\pi}{x}\right)} \leq \sqrt{x}e^{1} \\
\lim_{x\to 0^+} -\sqrt{x}e^{-1} &\leq \lim_{x\to 0^+} \sqrt{x}e^{\sin\left(\frac{\pi}{x}\right)} \leq \lim_{x\to 0^+} \sqrt{x}e^{1} \\
0 &\leq \lim_{x\to 0^+} \sqrt{x}e^{\sin\left(\frac{\pi}{x}\right)} \leq 0 \\
\lim_{x\to 0^+} \sqrt{x}e^{\sin\left(\frac{\pi}{x}\right)} &= 0
\end{aligned}$$

Example 12

Find

$$\lim_{x\to\infty} \frac{\sin(x)}{x^2+1}$$

Solution:

$$-1 \le \sin(x) \le 1$$

$$0 \le \sin^2(x) \le 1$$

$$0 \le \frac{\sin^2(x)}{x^2+1} \le \frac{1}{x^2+1}$$

$$\lim_{x\to\infty} 0 \le \lim_{x\to\infty} \frac{\sin^2(x)}{x^2+1} \le \lim_{x\to\infty} \frac{1}{x^2+1}$$

$$0 \le \lim_{x\to\infty} \frac{\sin^2(x)}{x^2+1} \le 0$$

$$\lim_{x\to\infty} \frac{\sin(x)}{x^2+1} = 0$$

Example 13

For which value(s) of k is the function defined by

$$f(x) = \begin{cases} \frac{\sin x}{x}, & x < 0 \\ 2e^{3x} - k, & x \geq 0 \end{cases}$$

continuous at $x = 0$?

Solution: We need to compute the left and right limit and we make them equal.

$$\lim_{x \to 0^-} f(x) = \lim_{x \to 0^-} \frac{\sin x}{x} = 1$$

$$\lim_{x \to 0^+} f(x) = \lim_{x \to 0^+} 2e^{3x} - k = 2 - k$$

we have $1 = 2 - k \to k = 1$.

Exercise 14

For which value(s) of k is the function defined by

$$f(x) = \begin{cases} \frac{\sin(2x)}{x}, & x < 0 \\ \cos x + x^2 + 4k, & x \geq 0 \end{cases}$$

continuous at $x = 0$?