Section 3.8
Derivative of the inverse function and logarithms
3 Lecture

Dr. Abdulla Eid

College of Science

MATHS 101: Calculus I

Topics

1. Inverse Functions (1 lecture).
2. Logarithms.
3. Derivative of the inverse function (1 lecture).
4. Logarithmic differentiation (1 lecture).

1 - Inverse functions (pre–calculus)

Definition 1

Let f be a function. The **inverse** function, denoted by f^{-1} of f is a *new* function such that

$$\underbrace{f}_{\text{outer}}(\underbrace{f^{-1}}_{\text{inner}}(x)) = x \text{ and } \underbrace{f^{-1}}_{\text{outer}}(\underbrace{f}_{\text{inner}}(x)) = x$$

(The function and its inverse cancel each other).

Example 2

(a) Let $f(x) = x + 5$, then $f^{-1}(x) = x - 5$ (we will see how to find the inverse shortly). Note that:

- $f(f^{-1}(x)) = f(x-5) = x - 5 + 5 = x$.
- $f^{-1}(f(x)) = f^{-1}(x+5) = x + 5 - 5 = x$.

(b) Let $f(x) = x^2 (x \geq 0)$, then $f^{-1}(x) = \sqrt{x}$ because:

- $f(f^{-1}(x)) = f(\sqrt{x}) = (\sqrt{x})^2 = x$.
- $f^{-1}(f(x)) = f^{-1}(x^2) = \sqrt{x^2} = |x| = x$.

Question: Does every function have an inverse? How to tell when a function has an inverse?

Answer: No, we use the **horizontal line test** if we have the graph of the function.

To find the inverse function

To find the inverse function

Algebraically

Step 1: Write $y = f(x)$.

Step 2: Switch x and y to get $x = f(y)$.

Step 3: Solve for y, i.e., isolate y alone to get $y = f^{-1}(x)$.

Geometrically

Step 1: Reflect the graph of $y = f(x)$ on the x–axis.

Step 2: rotate the resulting graph by 90° counterclockwise to get the graph of $f^{-1}(x)$.

Example 3

Find the inverse of $g(x) = 5x - 3$.

Solution:

Step 1: Write $y = g(x) \to y = 5x - 3$.

Step 2: Exchange x and y in step 1 $\to x = 5y - 3$.

Step 3: Solve the equation in step 1 for y

$$\begin{aligned} x &= 5y - 3 \\ x + 3 &= 5y \\ \frac{x+3}{5} &= y \end{aligned}$$

Hence we have

$$\boxed{g^{-1}(x) = \frac{x+3}{5}}$$

Exercise 4

Find the inverse function of

1. $f(x) = 3x + 2$.

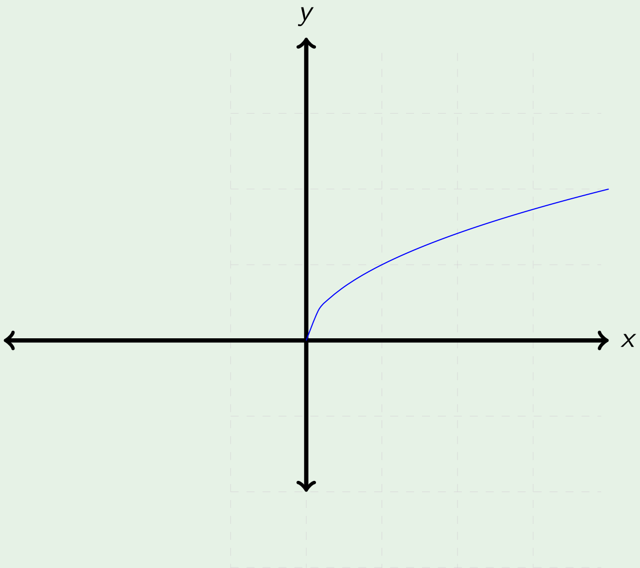
2. $f(x) = x^2 - 1 (x > 0)$.

3. $f(x) = \frac{1}{x}$.

4. $f(x) = \sqrt{x}$.

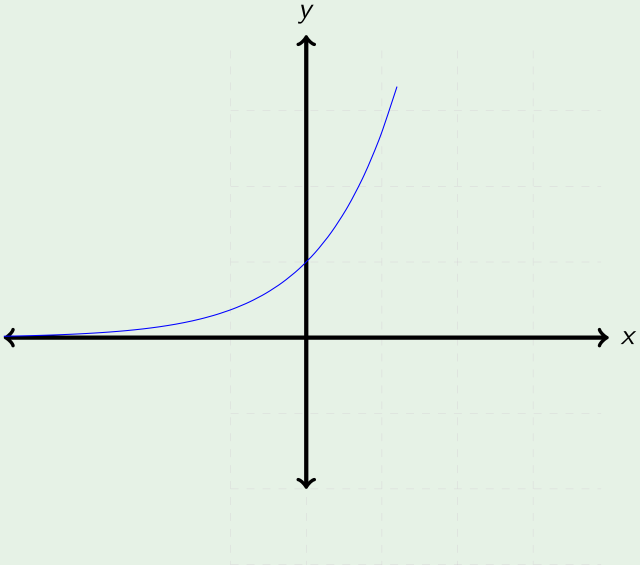
Example 5

Find the graph of the inverse function of the following functions:



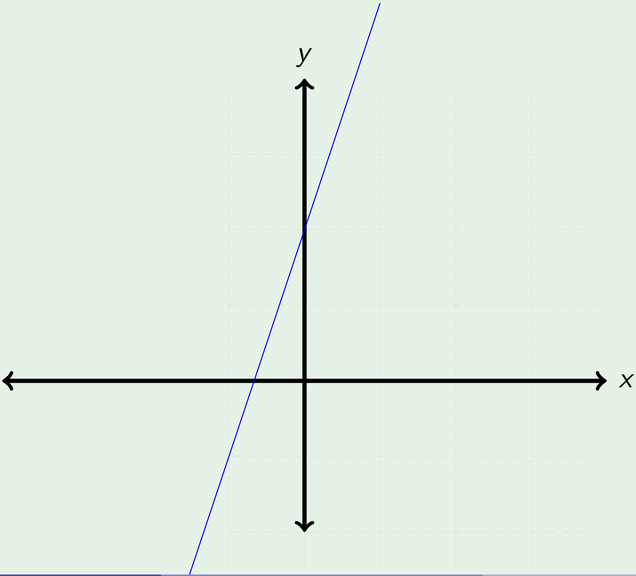
Example 6

Find the graph of the inverse function of the following functions:



Example 7

Find the graph of the inverse function of the following functions:



Inverse Trigonometric Functions

Example 8

Let $y = f(x) = \sin x$. Then the graph of the $f(x)$ is given by

Therefore, f has an inverse if $x \in \left[\frac{-\pi}{2}, \frac{\pi}{2}\right]$ and we write it as

$$f^{-1}(x) = \sin^{-1} x = \arcsin x.$$

1. Domain of $\sin^{-1}$ is $[-1, 1]$.
2. Range of $\sin^{-1}$ is $\left[\frac{-\pi}{2}, \frac{\pi}{2}\right]$.

Inverse Trigonometric Functions

Example 9

Let $y = f(x) = \cos x$. Then the graph of the $f(x)$ is given by

Therefore, f has an inverse if $x \in [0, \pi]$ and we write it as

$$f^{-1}(x) = \cos^{-1} x = \arccos x.$$

1. Domain of $\cos^{-1}$ is $[-1, 1]$.
2. Range of $\cos^{-1}$ is $[0, \pi]$.

Inverse Trigonometric Functions

Example 10

Let $y = f(x) = \tan x$. Then the graph of the $f(x)$ is given by Therefore,

f has an inverse if $x \in \left[\frac{-\pi}{2}, \frac{\pi}{2}\right]$ and we write it as

$$f^{-1}(x) = \tan^{-1} x = \arctan x.$$

1. Domain of $\tan^{-1}$ is $[-\infty, \infty$.
2. Range of $\tan^{-1}$ is $\left[\frac{-\pi}{2}, \frac{\pi}{2}\right]$.

Exercise 11

Find the domain, range, and the graph of inverse of the following functions:

1. $f(x) = \cot x$.
2. $f(x) = \sec x$.
3. $f(x) = \csc x$.