

Section 3.8

Derivative of the inverse function and logarithms

3 Lectures

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MATHS 101: Calculus I

Topics

- ① Inverse Functions (1 lecture).
- ② Logarithms.
- ③ Derivative of the inverse function (1 lecture).
- ④ Logarithmic differentiation (1 lecture).

Logarithmic Differentiation

Goal: To find the derivative of $y = f(x)$, where $f(x)$ is possibly involving quotient, product, powers, etc.

Example 1

$$① \quad y = \frac{(x+1)^4(3x^2+5)}{(4x-5)\sqrt[4]{4x^2+5}}.$$

$$② \quad y = \left(\frac{(x+5)(4x-2)^7}{x^2+5x+2} \right)^5.$$

$$③ \quad y = x^{\sqrt{x}}. \quad \text{---} \quad \text{variable}^{\text{variable}}.$$

$$④ \quad y = \ln x^{x^2+3x+5}.$$

Idea

To differentiate $y = f(x)$,

- 1 Take the natural logarithm of both sides to get

$$\ln y = \ln(f(x))$$

- 2 Simplify $\ln(f(x))$ by using the properties of the logarithms.
- 3 Differentiate both sides with respect to x .
- 4 Solve for y' .
- 5 Express the answer in terms of x (substitute $f(x)$ for y).

Example 2

Find y' for

$$y = x^{x+1}$$

Solution:

We take \ln of both sides to get and We simplify the right hand side using the properties of logarithms to get

$$\ln y = \ln (x^{x+1})$$

$$\ln y = (x + 1) \ln x$$

Now we differentiate both sides and we solve for y' .

$$\frac{1}{y} y' = (1)(\ln x) + (x + 1) \left(\frac{1}{x} \right)$$

$$y' = y \cdot \left(\ln x + \left(\frac{x + 1}{x} \right) \right)$$

$$y' = x^{x+1} \cdot \left(\ln x + \left(\frac{x + 1}{x} \right) \right)$$

Exercise 3

Find y' for

$$y = (x)^{\sin x}$$

Solution:

We take \ln of both sides to get and We simplify the right hand side using the properties of logarithms to get

$$\ln y = \ln ((x)^{\sin x})$$

$$\ln y = (\sin x) \ln(x)$$

Now we differentiate both sides and we solve for y' .

$$\frac{1}{y} y' = (\cos x)(\ln(x)) + (\sin x) \left(\frac{1}{x}\right)$$

$$y' = y \cdot \left((\cos x) \ln(x) + \left(\frac{\sin x}{x}\right) \right)$$

$$y' = (x)^{\sin x} \cdot \left((\cos x) \ln(x) + \left(\frac{\sin x}{x}\right) \right)$$

Example 4

Find y' for

$$y = (3x)^{\sqrt{x}}$$

Solution:

We take \ln of both sides to get and We simplify the right hand side using the properties of logarithms to get

$$\ln y = \ln \left((3x)^{\sqrt{x}} \right)$$

$$\ln y = \sqrt{x} \ln(3x)$$

Now we differentiate both sides and we solve for y' .

$$\frac{1}{y} y' = \left(\frac{1}{2\sqrt{x}} \right) (\ln(3x)) + (\sqrt{x}) \left(\frac{1}{3x} \cdot 3 \right)$$

$$y' = y \cdot \left(\frac{\ln(3x)}{2\sqrt{x}} + \left(\frac{\sqrt{x}}{x} \right) \right)$$

$$y' = (3x)^{\sqrt{x}} \cdot \left(\frac{\ln(3x)}{2\sqrt{x}} + \left(\frac{\sqrt{x}}{x} \right) \right)$$

Exercise 5

Find y' for

$$y = (\ln x)^{\ln x}$$

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Example 6

(General Form) Find y' for

$$y = f(x)^{g(x)}$$

Solution:

We take \ln of both sides to get and We simplify the right hand side using the properties of logarithms to get

$$\ln y = \ln \left(f(x)^{g(x)} \right)$$

$$\ln y = g(x) \ln f(x)$$

Now we differentiate both sides and we solve for y' .

$$\frac{1}{y} y' = (g'(x))(\ln f(x)) + (g(x)) \left(\frac{f'(x)}{f(x)} \right)$$

$$y' = y \cdot \left[(g'(x))(\ln f(x)) + (g(x)) \left(\frac{f'(x)}{f(x)} \right) \right]$$

$$y' = f(x)^{g(x)} \cdot \left[(g'(x))(\ln f(x)) + (g(x)) \left(\frac{f'(x)}{f(x)} \right) \right]$$

Example 7

Find y' for

$$y = \sqrt{\frac{5-4x}{1+x^2}}$$

Solution:

We take \ln of both sides to get

$$\ln y = \ln \left(\sqrt{\frac{5-4x}{1+x^2}} \right)$$

We simplify the right hand side using the properties of logarithms to get

$$\ln y = \ln \left(\sqrt{\frac{5-4x}{1+x^2}} \right)$$

$$\ln y = \ln \left(\frac{5-4x}{1+x^2} \right)^{\frac{1}{2}} = \frac{1}{2} \ln \left(\frac{5-4x}{1+x^2} \right)$$

$$\ln y = \frac{1}{2} (\ln(5-4x) - \ln(1+x^2))$$

Continue...

Now we differentiate both sides and we solve for y' .

$$\ln y = \frac{1}{2} (\ln(5 - 4x) - \ln(1 + x^2))$$

$$\frac{1}{y} y' = \frac{1}{2} \left(\frac{-4}{5 - 4x} - \frac{2x}{1 + x^2} \right)$$

$$y' = y \cdot \frac{1}{2} \left(\frac{-4}{5 - 4x} - \frac{2x}{1 + x^2} \right)$$

$$y' = \sqrt{\frac{5 - 4x}{1 + x^2}} \cdot \frac{1}{2} \left(\frac{-4}{5 - 4x} - \frac{2x}{1 + x^2} \right)$$

Exercise 8

Find y' for

$$y = \frac{(1 - 2x)^3(4 + 5x^6)^7}{\sqrt[3]{8 - 9x}}$$

Solution:

We take \ln of both sides to get

$$\ln y = \ln \left(\frac{(1 - 2x)^3(4 + 5x^6)^7}{\sqrt[3]{8 - 9x}} \right)$$

We simplify the right hand side using the properties of logarithms to get

$$\begin{aligned}\ln y &= \ln \left(\frac{(1 - 2x)^3(4 + 5x^6)^7}{\sqrt[3]{8 - 9x}} \right) \\ \ln y &= \ln(1 - 2x)^3 + \ln(4 + 5x^6)^7 - \ln \sqrt[3]{8 - 9x} \\ &= 3 \ln(1 - 2x) + 7 \ln(4 + 5x^6) - \frac{1}{3} \ln(8 - 9x)\end{aligned}$$

Continue...

Now we differentiate both sides and we solve for y' .

$$\ln y = 3 \ln(1 - 2x) + 7 \ln(4 + 5x^6) - \frac{1}{3} \ln(8 - 9x)$$

$$\frac{1}{y} y' = 3 \frac{-2}{1 - 2x} + 7 \frac{30x^5}{4 + 5x^6} - \frac{-9}{3(8 - 9x)}$$

$$y' = y \cdot \left(\frac{-6}{1 - 2x} + \frac{210x^5}{4 + 5x^6} - \frac{-9}{3(8 - 9x)} \right)$$

$$y' = \frac{(1 - 2x)^3 (4 + 5x^6)^7}{\sqrt[3]{8 - 9x}} \cdot \left(\frac{-6}{1 - 2x} + \frac{210x^5}{4 + 5x^6} - \frac{-9}{3(8 - 9x)} \right)$$