

Section 3.9

Inverse Trigonometric Functions

1 Lecture

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MATHS 101: Calculus I

Inverse Trigonometric functions

- ① The derivative of the **basic** inverse trigonometric functions.
- ② The derivative of functions that involve inverse trigonometric functions.
- ③ Identities of inverse trigonometric functions using differential calculus.

$f(x) = \sin x$ has an *inverse* if $x \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$. The inverse of the sine function is denoted by

$$f^{-1}(x) = \sin^{-1} x \quad (= \arcsin x)$$

with the following properties:

- 1 The graph of $y = \sin^{-1} x$ is:
- 2 The domain of $\sin^{-1} x$ is $[-1, 1]$
- 3 The range of $\sin^{-1} x$ is $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$
- 4 $\sin(\sin^{-1} x) = x$
- 5 $\sin^{-1}(\sin x) = x$

Exercise 1

Write a similar definition for the $\tan^{-1} x$ and $\sec^{-1} x$ and explore their properties.

Derivative of $\sin^{-1} x$

Example 2

Find $\frac{d}{dx} (\sin^{-1} x)$.

Solution:

$$y = \sin^{-1} x, \text{ find } y'$$

$$\sin y = \sin (\sin^{-1} x)$$

$$\sin y = x$$

$$\cos y \cdot y' = 1$$

$$y' = \frac{1}{\cos y} = \sec y$$

$$\sin y = x = \frac{x}{1} = \frac{\text{opp}}{\text{hypo}}$$

$$(\text{adj})^2 + (\text{opp})^2 = (\text{hypo})^2$$

$$(\text{adj})^2 + x^2 = 1$$

$$(\text{adj})^2 = 1 - x^2$$

$$\text{adj} = \sqrt{1 - x^2}$$

$$\sec y = \frac{\text{hypo}}{\text{adj}} = \frac{1}{\sqrt{1 - x^2}}$$

$$y' = \frac{d}{dx} (\sin^{-1} x) = \frac{1}{\sqrt{1 - x^2}}$$

Exercise 3

Derive a formula for $\frac{d}{dx} (\cos^{-1} x)$

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Derivative of $\tan^{-1} x$

Example 4

Find $\frac{d}{dx} (\tan^{-1} x)$.

Solution:

$$y = \tan^{-1} x, \text{ find } y'$$

$$\tan y = \tan (\tan^{-1} x)$$

$$\tan y = x$$

$$\sec^2 y \cdot y' = 1$$

$$y' = \frac{1}{\sec^2 y} = \cos^2 y$$

$$\tan y = x = \frac{x}{1} = \frac{\text{opp}}{\text{adj}}$$

$$(\text{adj})^2 + (\text{opp})^2 = (\text{hypo})^2$$

$$1 + x^2 = (\text{hypo})^2$$

$$(\text{hypo})^2 = 1 + x^2$$

$$\text{hypo} = \sqrt{1 + x^2}$$

$$\cos y = \frac{\text{adj}}{\text{hypo}} = \frac{1}{\sqrt{1 + x^2}}$$

$$y' = \frac{d}{dx} (\tan^{-1} x) = \frac{1}{1 + x^2}$$

Exercise 5

Derive a formula for $\frac{d}{dx} (\cot^{-1} x)$

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Derivative of $\sec^{-1} x$

Example 6

Find $\frac{d}{dx} (\sec^{-1} x)$.

Solution:

$$y = \sec^{-1} x, \text{ find } y'$$

$$\tan y = \sec (\sec^{-1} x)$$

$$\sec y = x$$

$$\sec y \tan y \cdot y' = 1$$

$$y' = \frac{1}{\sec y \tan y} = \cos y \cot y$$

$$\sec y = x = \frac{x}{1} = \frac{\text{hypo}}{\text{adj}}$$

$$(\text{adj})^2 + (\text{opp})^2 = (\text{hypo})^2$$

$$1 + (\text{opp})^2 = x^2$$

$$(\text{opp})^2 = x^2 - 1$$

$$\text{opp} = \sqrt{x^2 - 1}$$

$$\cot y = \frac{\text{adj}}{\text{opp}} = \frac{1}{\sqrt{x^2 - 1}}$$

$$y' = \frac{d}{dx} (\sec^{-1} x) = \frac{1}{|x| \sqrt{x^2 - 1}}$$

Exercise 7

Derive a formula for $\frac{d}{dx} (\csc^{-1} x)$

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Summary

(Derivative of the Inverse Trigonometric functions)

$$(\sin^{-1} x)' = \frac{1}{\sqrt{1-x^2}}$$

$$(\cos^{-1} x)' = \frac{-1}{\sqrt{1-x^2}}$$

$$(\tan^{-1} x)' = \frac{1}{1+x^2}$$

$$(\cot^{-1} x)' = \frac{-1}{1+x^2}$$

$$(\sec^{-1} x)' = \frac{1}{|x|\sqrt{x^2-1}}$$

$$(\csc^{-1} x)' = \frac{-1}{|x|\sqrt{x^2-1}}$$

2 - Derivative of functions that involve the inverse trigonometric functions

Example 8

Find y' if $y = \tan^{-1}(\sqrt{x})$.

Solution: We have $y = \tan^{-1}(\sqrt{x})$

$$y' = \frac{1}{1 + (\sqrt{x})^2} \cdot \left(\frac{1}{2\sqrt{x}} \right)$$

Exercise 9

Find y' if $y = \sqrt{\tan^{-1} x}$.

Solution: We have $y = \sqrt{\tan^{-1} x}$

$$y' = \frac{1}{2\sqrt{\tan^{-1} x}} \cdot \left(\frac{1}{1+x^2} \right)$$

Example 10

Find y' if $y = \cos^{-1}(e^x)$.

Solution: We have $y = \cos^{-1}(e^x)$

$$y' = \frac{-1}{\sqrt{1 - (e^x)^2}} \cdot (e^x)$$

Exercise 11

Find y' if $y = \arcsin(3 - 2x)$.

Solution: We have $y = \arcsin(3 - 2x)$

$$y' = \frac{1}{\sqrt{1 - (3 - 2x)^2}} \cdot (-2)$$

Example 12

Find y' if $y = \sin^{-1}(\sqrt{\sin x})$.

Solution: We have $y = \sin^{-1}(\sqrt{\sin x})$

$$\begin{aligned}y' &= \frac{1}{\sqrt{1 - (\sqrt{\sin x})^2}} \cdot (\sqrt{\sin x})' \\&= \frac{1}{\sqrt{1 - (\sqrt{\sin x})^2}} \cdot \frac{1}{2\sqrt{\sin x}} \cdot \cos x \\&= \frac{1}{\sqrt{1 - \sin x}} \cdot \frac{1}{2\sqrt{\sin x}} \cdot \cos x\end{aligned}$$

Exercise 13

Find y' if $y = x \sin^{-1} x + \sqrt{1 - x^2}$ and simplify your answer.

Solution:

$$\begin{aligned}y' &= (1) \sin^{-1} x + x \cdot \frac{1}{\sqrt{1-x^2}} + \frac{1}{2\sqrt{1-x^2}} \cdot (-2x) \\y' &= \sin^{-1} x + \frac{x}{\sqrt{1-x^2}} - \frac{x}{\sqrt{1-x^2}} \\&= \sin^{-1} x\end{aligned}$$

Identities of inverse trigonometric functions using differential calculus.

Example 14

Find y' if $y = \sin^{-1} + \cos^{-1} x$.

Solution:

$$y' = \frac{1}{\sqrt{1-x^2}} + \frac{-1}{\sqrt{1-x^2}}$$
$$y' = 0$$

The derivative is always zero, which means the function y is a constant.

$$y = \sin^{-1} + \cos^{-1} x = C \rightarrow \sin^{-1}(0) + \cos^{-1}(0) = C \rightarrow \frac{\pi}{2} = C$$

Hence the identity is

$$\boxed{\sin^{-1} + \cos^{-1} x = \frac{\pi}{2}}$$

Exercise 15

Show using calculus that

$$\textcircled{1} \cot^{-1} x = \frac{\pi}{2} - \tan^{-1} x$$

$$\textcircled{2} \csc^{-1} x = \frac{\pi}{2} - \sec^{-1} x$$

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Example 16

Find y' if $y = \tan^{-1} x + \tan^{-1} \left(\frac{1}{x}\right)$.

Solution:

$$y' = \frac{1}{1+x^2} + \frac{1}{1+\left(\frac{1}{x}\right)^2} \cdot \frac{-1}{x^2} \rightarrow \frac{1}{1+x^2} + \frac{x^2}{1+x^2} \cdot \frac{-1}{x^2}$$
$$y' = \frac{1}{1+x^2} - \frac{1}{1+x^2} \rightarrow y' = 0$$

The derivative is always zero, which means the function y is a constant.

$$y = \tan^{-1} x + \tan^{-1} \left(\frac{1}{x}\right) = C \rightarrow \tan^{-1}(1) + \tan^{-1} \left(\frac{1}{1}\right) = C \rightarrow \frac{\pi}{2} = C$$

Hence the identity is

$$\tan^{-1} x + \tan^{-1} \left(\frac{1}{x}\right) = \frac{\pi}{2}$$

Exercise 17

Show that $\sec^{-1}(-x) + \sec^{-1}(x) = \pi$.

Exercise 18

(At home) Show that

$$\tan^{-1} x + \tan^{-1} a = \tan^{-1} \left(\frac{x + a}{1 - ax} \right)$$

and use it to compute $\tan^{-1} 1 + \tan^{-1} 2 + \tan^{-1} 3$.

Exercise 19

(Challenging problem) What is wrong in

$$\frac{d}{dx} (\sin^{-1} x + \sec^{-1} x) = \frac{1}{\sqrt{1-x^2}} + \frac{1}{|x|\sqrt{x^2-1}}$$