Section 4.3
Concavity and Curve Sketching
1.5 Lectures

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MATHS 101: Calculus I

Concavity

Increasing Function has three cases

Question: How to distinguish between these three types of behavior?
Answer: Recall: If $g(x)$ is increasing, then $g'(x) > 0$.

1. If $f''(x) > 0$, then the curve is concave upward (CU).
2. If $f''(x) < 0$, then the curve is concave downward (CD).
3. If $f''(x) = 0$ (for all x), then $f(x)$ has no curvature (line).

Inflection Points

Definition 1

A number c is called an inflection point of $f(x)$ if at these point, the function changes from concave upward to downward and vice verse. The candidates are the points c, where

$$f''(c) = 0 \text{ or } f''(c) \text{ does not exist}$$

Example 2

Discuss the following curve with respect to concavity and inflection points.

$$f(x) = x^3$$

Solution:

We find the derivatives first which are

$$f'(x) = 3x^2$$
$$f''(x) = 6x$$

To find the inflection points, we find where the second derivative equal to zero or does not exist.

$$f''(x) = 0$$
$$\text{numerator} = 0$$
$$6x = 0$$
$$x = 0$$

$$f'(x) \text{ does not exist}$$
$$\text{denominator} = 0$$
$$1 = 0$$

Always False

No Solution

Number Line

1. f is CD in $(-\infty, 0)$.
2. f is CU in $(0, \infty)$.
3. f has inflection point at $x = 0$ with value $f(0) = 0$. As a point it is $(0, 0)$

Exercise 3

Discuss the following curve with respect to concavity and inflection points.

$$f(x) = x^4 - 3x^3 + 3x^2 - 5$$

Solution:

We find the derivatives first which are

$$f'(x) = 4x^3 - 9x^2 + 6x$$
$$f''(x) = 12x^2 - 18x + 6$$

$$f''(x) = 0$$
$$\text{numerator} = 0$$
$$12x^2 - 18x + 6 = 0$$
$$x = 1 \text{ or } x = \frac{1}{2}$$

$$f'(x) \text{ does not exist}$$
$$\text{denominator} = 0$$
$$1 = 0$$

Always False

No Solution

Number Line

1. f is CD in $(\frac{1}{2}, 1)$.
2. f is CU in $(-\infty, \frac{1}{2}) \cup (1, \infty)$.
3. f has inflection point at $x = \frac{1}{2}$ with value $f(\frac{1}{2}) =$ and at $x = 1$ with value $f(1) =$.

Exercise 4

Discuss the following curve with respect to concavity and inflection points.

$$f(x) = 2 + \sin x, \qquad x \in [0, 2\pi]$$

Solution:

We find the derivatives first which are

$$f'(x) = \cos x$$
$$f''(x) = -\sin x$$

To find the inflection points, we find where the second derivative equal to zero or does not exist.

$$f''(x) = 0$$
$$\text{numerator} = 0$$
$$\sin x = 0$$
$$x = 0 \text{ or } x = \pi$$

$$f''(x) = \text{ does not exist}$$
$$\text{denominator} = 0$$
$$-1 = 0$$

Always False

No Solution

Number Line

1. f is CU in $(\pi, 2\pi)$.
2. f is CD in $(0, \pi)$.
3. f has an inflection point at $(\pi, 0)$.

Example 5

(All in All) Find the intervals where the function is increasing/decreasing, concave upward, concave downward, find all local max/min, find inflection points and sketch the graph of the function.

$$f(x) = x^3 - 6x^2 + 9x + 1$$

Solution: We find the derivative first which is

$$f'(x) = 3x^2 - 12x + 9$$
$$f''(x) = 6x - 12$$

To find the critical points, we find where the derivative equal to zero or does not exist.

$$f'(x) = 0$$
$$\text{numerator} = 0$$
$$3x^2 - 12x + 9 = 0$$
$$x = 1 \text{ or } x = 3$$

$$f'(x) \text{ does not exist}$$
$$\text{denominator} = 0$$
$$1 = 0$$

Always False

No Solution

Number Line

1. f is increasing in $(-\infty, 1) \cup (3, \infty)$.
2. f is decreasing in $(1, 3)$.
3. f has a local maximum at $x = 1$ with value $f(1) = 5$.
4. f has a local minimum at $x = 3$ with value $f(3) = 1$.

Recall that the derivatives are

$$f'(x) = 3x^2 - 12x + 9$$
$$f''(x) = 6x - 12$$

To find the inflection points, we find where the second derivative equal to zero or does not exist.

$$f''(x) = 0$$
$$\text{numerator} = 0$$
$$6x - 12 = 0$$
$$x = 2$$

$f'(x)$ does not exist

$$\text{denominator} = 0$$
$$1 = 0$$

Always False

No Solution

Number Line

1. f is CD in $(-\infty, 2)$.
2. f is CU in $(2, \infty)$.
3. f has inflection point at $x = 2$ with value $f(2) = 3$.

Exercise 6

(All in All) Find the intervals where the function is increasing/decreasing, concave upward, concave downward, find all local max/min, find inflection points and sketch the graph of the function.

$$f(x) = x^5 - 4x^4$$

Solution: We find the derivative first which is

$$f'(x) = 5x^4 - 20x^3$$
$$f''(x) = 20x^3 - 60x^2$$

To find the critical points, we find where the derivative equal to zero or does not exist.

$$f'(x) = 0$$
$$\text{numerator} = 0$$
$$5x^4 - 20x^3 = 0$$
$$x = 0 \text{ or } x = 4$$

$$f'(x) \text{ does not exist}$$
$$\text{denominator} = 0$$
$$1 = 0$$

Always False

No Solution

Recall that the derivatives are

$$f'(x) = 5x^4 - 20x^3$$
$$f''(x) = 20x^3 - 60x^2$$

To find the inflection points, we find where the second derivative equal to zero or does not exist.

$$f''(x) = 0$$
$$\text{numerator} = 0$$
$$20x^3 - 60x^2 = 0$$
$$x = 1 \text{ or } x = 3$$

$$f'(x) \text{ does not exist}$$
$$\text{denominator} = 0$$
$$1 = 0$$

Always False

No Solution

Number Line

1. f is CD in $(-\infty, 3)$.
2. f is CU in $(3, \infty)$.
3. f has inflection point at $x = 3$ with value $f(3) =$.

Example 7

(All in All) Find the intervals where the function is increasing/decreasing, concave upward, concave downward, find all local max/min, find inflection points and sketch the graph of the function.

$$f(x) = x - \sin x, \qquad x \in [0, 4\pi]$$

Solution: We find the derivative first which is

$$f'(x) = 1 - \cos x$$
$$f''(x) = -\sin x$$

$f'(x) = 0$

numerator $= 0$

$1 - \cos x = 0 \rightarrow \cos x = 1$

$x = 0$ or $x = 2\pi$ or $x = 4\pi$

$f'(x)$ does not exist

denominator $= 0$

$1 = 0$

Always False

No Solution

Number Line

1. f is increasing in $(0, 2\pi) \cup (2\pi, 4\pi)$.
2. f is not decreasing on $[0, 4\pi]$.
3. f has no local maximum nor local minimum.

Recall that the derivatives are

$$f'(x) = 1 - \cos x$$
$$f''(x) = \sin x$$

$$f''(x) = 0$$
$$\text{numerator} = 0$$
$$\sin x = 0$$
$$x = 0 \text{ or } x = \pi \text{ or } x = 2\pi$$
$$\text{or } x = 2\pi \text{ or } 4\pi$$

$$f'(x) \text{ does not exist}$$
$$\text{denominator} = 0$$
$$1 = 0$$

Always False

No Solution

Number Line

1. f is CD in $(0, \pi) \cup (2\pi, 3\pi)$.
2. f is CU in $(\pi, 2\pi) \cup (3\pi, 4\pi)$.
3. f has inflection point at $x = \pi, 2\pi, 3\pi$ with points (π, π), $(2\pi, 2\pi)$, and $(3\pi, 3\pi)$.

Number Line

1. f is increasing in $(-\infty, 0) \cup (4, \infty)$.
2. f is decreasing in $(0, 4)$.
3. f has a local maximum at $x = 0$ with value $f(0) = 0$.
4. f has a local minimum at $x = 4$ with value $f(4) =$.

General Guidelines to sketch the curve of a rational function $y = f(x)$

1. Find the domain of $f(x)$.
2. Find the x–intercept by solving $y = 0$ (whenever possible). Find the y–intercept by setting $x = 0$ to find $y = f(0)$ (if possible).
3. Find the horizontal and vertical asymptotes.
4. Find the local maximum, local minimum, inflection points the interval where the function is increasing, decreasing, concave upward, concave downward.

Example 8

Sketch the function

$$y = f(x) = \frac{x}{x-2}$$

Solution: 1- The domain of $f(x)$ is all real numbers except $x = 2$.
2- x–intercept: solve

$$y = 0 \to \frac{x}{x-2} = 0 \to x = 0$$

so $(0, 0)$ is the x–intercept. Moreover, it is the y–intercept.
3- Horizontal Asympotote:

$$\lim_{x\to\infty} f(x) = \lim_{x\to\infty} \frac{x}{x-2} = 1$$

So $y = 1$ is the horizontal asymptote. Moreover, $x = 2$ is the vertical asymptote.

4- Local max, local min

Example 9

Sketch the function

$$y = f(x) = \frac{x}{x-2}$$

Solution: We find the derivative first which is

$$f'(x) = \frac{-2}{(x-2)^2}$$

$$f''(x) = \frac{4}{(x-2)^3}$$

To find the critical points, we find where the derivative equal to zero or does not exist.

$f'(x)$ does not exist

$$\text{numerator} = 0$$

$$-2 = 0$$

Always False

$$f'(x) = 0$$

$$\text{denominator} = 0$$

$$x - 2 = 0$$

Number Line

1. f is decreasing in $(-\infty, 2) \cup (2, \infty)$.
2. f has no local maximum or local minimum.

Recall that the derivatives are

$$f'(x) = \frac{-2}{(x-2)^2}$$
$$f''(x) = \frac{4}{(x-2)^3}$$

$f''(x)$ does not exist

$$\text{numerator} = 0$$
$$4 = 0$$

Always False

No Solution

$f''(x) = 0$

$$\text{denominator} = 0$$
$$x - 2 = 0$$
$$x = 2$$

Number Line

1. f is CD in $(-\infty, 2)$.
2. f is CU in $(2, \infty)$.
3. f has no inflection point at $x = 2$ since it is not in the domain of $f(x)$.

Exercise 10

Sketch the function

$$y = f(x) = \frac{4x}{x^2 - 1}$$