Section 5.4
Fundamental Theorem of Calculus
2 Lectures

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MATHS 101: Calculus 1

Definite Integral

Recall: The integral is used to find area under the curve over an interval $[a, b]$

Idea: To cover the area by as many rectangles as possible and then we will get better and better estimate if we increase the number of rectangles.

Question: When will we get an exact estimate for the area?

Answer: When the number of rectangle $\to \infty$. In that case, we write the area by

$$\text{Area} = \int_a^b f(x)\,dx$$

This integral is called definite integral. The number a and b are called the *lower limit and upper limit of integration* respectively.

The Fundamental Theorem of Calculus, Part 1

Question: What is the relation between definite integral (finding the area) and the indefinite integral (finding the anti-derivative)?

Answer: The **Fundamental Theorem of Calculus**. One of the great achievement of the human mind.

We will focus on the function

$$g(x) = \int_0^x f(t)\,dt \text{ --- '' area so far ``}$$

Example 1

Let $g(x) = \int_0^x f(t)\,dt$. Find the following:

1. $g(0)$
2. $g(1)$
3. $g(2)$
4. $g(3)$
5. $g(4)$

The Fundamental Theorem of Calculus, Part 1

Question: What is the derivative of $g(x) = \int_0^x f(t)\,dt$?

Theorem 2

$$\frac{d}{dx}\left(\int_0^x f(t)\,dt\right) = f(x)$$

In general,

Theorem 3

$$\frac{d}{dx}\left(\int_{g(x)}^{h(x)} f(t)\,dt\right) = f(h(x))(h(x))' - f(g(x))(g(x))'$$

Example 4

Find the derivative of $g(x) = \int_1^x \frac{1}{t^3+1}\, dt$.

Solution:

$$g'(x) = \left(\frac{1}{x^3+1}\right)(x)' - \left(\frac{1}{(1)^3+1}\right)(1)'$$
$$= \frac{1}{x^3+1}$$

Exercise 5

Find the derivative of $g(x) = \int_x^2 \sqrt{1+\sec t}\, dt$.

Solution:

$$
\begin{aligned}
g'(x) &= (\sqrt{1+\sec 2})(2)' - (\sqrt{1+\sec x})(x)' \\
&= -\sqrt{1+\sec x}
\end{aligned}
$$

Example 6

Find the derivative of $g(x) = \int_1^{\tan x} \sqrt{t + \sqrt{t}}\, dt$.

Solution:

$$g'(x) = \left(\sqrt{\tan x + \sqrt{\tan x}}\right)(\tan x)' - \left(\sqrt{1 + \sqrt{1}}\right)(1)'$$
$$= \left(\sqrt{\tan x + \sqrt{\tan x}}\right)\sec^2 x$$

Example 7

Find the derivative of $g(x) = \int_{\sec x}^{x^3} e^t + 5t^2\, dt$.

Solution:

$$\begin{aligned} g'(x) &= \left(e^{x^3} + 5(x^3)^2\right)\left(x^3\right)' - \left(e^{\sec x} + 5(\sec x)^2\right)\left(\sec x\right)' \\ &= \left(e^{x^3} + 5x^6\right)\left(3x^2\right) - \left(e^{\sec x} + 5\sec^2 x\right)\left(\sec x \tan x\right) \end{aligned}$$

Exercise 8

Find the derivative of $g(x) = \int_{\sin x}^{x^9} \tan^9 t \, dt$.

The Fundamental Theorem of Calculus, Part 2

Question: How to evaluate the definite integral?

Theorem 9

If f is continuous on the interval $[a, b]$ and F is the anti-derivative of f, then

$$\int_a^b f(x)\,dx = \Big[\underbrace{F(x)}_{antiderivative}\Big]_a^b = F(b) - F(a)$$

1. Definite integral $\int_a^b f(x)\,dx$ gives a **number** representing the area.
2. Indefinite integral $\int f(x)\,dx$ gives a **function**.

Example 10

Find $\int_{-1}^{2}(x^3 - 6x)\,dx$.

Solution: [1]

$$\int_{-1}^{2}(x^3 - 6x)\,dx = \left[\frac{1}{4}x^4 - 3x^2\right]_{-1}^{2}$$
$$\left(\frac{1}{4}(2)^4 - 3(2)^2\right) - \left(\frac{1}{4}(-1)^4 - 3(-1)^2\right) = \frac{-21}{4}$$

[1] Direct evaluation

Exercise 11

Find $\int_1^4 \left(x^3 + \frac{1}{x}\right)\, dx$.

Example 12

Find $\int_1^9 6\sqrt{x}\,dx$.

Solution: [2]

$$\int_1^9 6\sqrt{x}\,dx = \int_1^9 6x^{\frac{1}{2}}\,dx = \left[6\frac{2}{3}x^{\frac{3}{2}}\right]_1^9$$

$$\left(4(9)^{\frac{3}{2}}\right) - \left(4(1)^{\frac{3}{2}}\right) = 104$$

[2] Direct evaluation

Exercise 13

Find $\int_{-1}^{1}(x+1)^2\,dx$.

Example 14

Find $\int_1^2 \frac{x^5+3x^3}{x^4}\,dx$.

Solution: [3]

$$\int_1^2 \frac{x^5+3x^3}{x^4}\,dx = \int_1^2 x+\frac{3}{x}\,dx = \left[\frac{1}{2}x^2+3\ln|x|\right]_1^2$$

$$\left(\frac{1}{2}(2)^2+3\ln 2\right)-\left(\frac{1}{2}(1)^2+3\ln 1\right) = \frac{3}{2}+3\ln 2$$

[3] Direct evaluation

Exercise 15

Find $\int_0^1 \frac{2}{1+x^2}\,dx$.

Solution: [4]

$$\int_0^1 \frac{2}{1+x^2}\,dx = \left[2\tan^{-1} x\right]_0^1$$

$$\left(2\tan^{-1} 1\right) - \left(2\tan^{-1} 0\right) = \frac{\pi}{2}$$

[4] Direct evaluation

Example 16

If $\int_a^2 (x+1)^2\,dx = 9$, then find the value of a.

Solution: [5]

$$9 = \int_a^2 (x+1)^2\,dx = \int_a^2 (x^2+2x+1)\,dx = \left[\frac{1}{3}x^3 + x^2 + x\right]_a^2$$

$$9 = \left(\frac{26}{3}\right) - \left(\frac{1}{3}a^3 + a^2 + a\right)$$

$$9 = -\frac{1}{3}a^3 - a^2 - a + \frac{26}{3}$$

$$0 = -\frac{1}{3}a^3 - a^2 - a + \frac{-2}{3}$$

$$a = -2$$

[5] Finding limit of integration

Exercise 17

If $\int_a^3 (3x^2 + 2x)\, dx = 36$, then find the value of a.

Solution: [6]

$$
\begin{aligned}
36 &= \int_a^2 (3x^2 + 2x)\, dx = \int_a^3 (3x^2 + 2x)\, dx = \left[x^3 + x^2\right]_a^3 \\
36 &= (36) - \left(a^3 + a^2\right) \\
36 &= -a^3 - a^2 + 36 \\
0 &= -a^3 - a^2 \\
0 &= -a^2(a + 1) \\
a &= 0 \text{ or } a = -1
\end{aligned}
$$

[6] Finding limit of integration

Properties of Integration

Recall: Definite integrals compute the area under the curve, i.e.,

$$\text{Area} = \int_a^b f(x)\,dx$$

1. $\int_a^b [c \cdot f(x)]\,dx = c \cdot \int_a^b f(x)\,dx.$
2. $\int_a^b [f(x) + g(x)]\,dx = \int_a^b f(x)\,dx + \int_a^b g(x)\,dx.$
3. $\int_a^a f(x)\,dx = 0.$
4. $\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx.$
5. $\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b g(x)\,dx.$
6. If $f(x) \leq g(x)$ on $[a, b]$, then $\int_a^b f(x)\,dx \leq \int_a^b g(x)\,dx.$

Example 18

If $\int_0^2 f(x)\,dx = 3$, $\int_0^2 g(x)\,dx = 2$, then find $\int_0^2 [4f(x) + g(x)]\,dx$.

Solution: [7]

$$\begin{aligned}\int_0^2 [4f(x) + g(x)] &= 4\int_0^2 [f(x)\,dx] + \int_0^2 [g(x)]\,dx \\ &= 4(3) + 2 \\ &= 14\end{aligned}$$

[7] Properties of integral

Exercise 19

If $\int_1^5 f(x)\,dx = 3$, $\int_1^3 f(x)\,dx = 1$, and $\int_1^3 h(x)\,dx = 5$ then find [a]

1. $\int_1^5 -2f(x)\,dx$.
2. $\int_1^3 [f(x) + h(x)]\,dx$.
3. $\int_1^3 [2f(x) - 5h(x)]\,dx$.
4. $\int_5^1 f(x)\,dx$.
5. $\int_3^5 f(x)\,dx$.
6. $\int_3^1 [h(x) - f(x)]\,dx$.
7. $\int_3^3 [h(x) - f(x)]\,dx$.

[a] Properties of integral

Example 20

Given

$$f(x) = \begin{cases} 4x+2, & x < 2 \\ 3x^2 - 2, & 2 \le x < 6 \\ 106, & x \ge 6 \end{cases}$$

Evaluate $\int_0^4 f(x)\,dx$

Solution: [8]

$$\begin{aligned} \int_0^4 f(x)\,dx &= \int_0^2 f(x)\,dx + \int_2^4 f(x)\,dx \\ &= \int_0^2 4x+2\,dx + \int_2^4 3x^2 - 2\,dx \\ &= \left[2x^2 + 2x\right]_0^2 + \left[x^3 - 2x\right]_2^4 \\ &= 16 + 56 - 4 = 68 \end{aligned}$$

[8] Properties of integration

Exercise 21

Given

$$f(x) = \begin{cases} 3x^2, & x < 1 \\ 2x + 1, & 1 \leq x \end{cases}$$

Evaluate $\int_{-1}^{2} f(x)\, dx$

Exercise 22

Evaluate $\int_{-5}^{7} |x|\, dx$