Section 1.5
Inverse of a matrix

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MATHS 211: Linear Algebra

Goal:

1. To define the inverse of a matrix.
2. To find the inverse of a matrix.
3. To explore some of the properties of the inverse of matrices.

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1 - Definition of the inverse of a matrix

Recall:

- If a is a real number, then the **additive inverse** of a is $-a$, such that

$$a + (-a) = 0 \text{ and } (-a) + a = 0$$

- If a is a nonzero real number, then the **multiplicative inverse** of a is $\frac{1}{a}$, such that

$$a \cdot \frac{1}{a} = 1 \text{ and } \frac{1}{a} \cdot a = 1$$

- If f is a function passing the horizontal line test, then the inverse of f is f^{-1} such that

$$(f \circ f^{-1})(x) = x \text{ and } (f^{-1} \circ f)(x) = x$$

Definition 1

Let A be an $n \times n$–matrix. The **inverse matrix** (if it exists) of A is another matrix A^{-1} such that

$$A \cdot A^{-1} = I_n \text{ and } A^{-1} \cdot A = I_n$$

2 - How to find the inverse of a matrix

We write

$$(A|I_n) \tag{1}$$

and then we reduce (1) to get,

$$(I_n|A^{-1})$$

Note: If we can't reduce (1), then the matrix has no inverse.

Example 2

Find A^{-1} for

$$A = \begin{pmatrix} 3 & 1 \\ 4 & 1 \end{pmatrix}$$

Solution:

$$\left(\begin{array}{cc|cc} 3 & 1 & 1 & 0 \\ 4 & 1 & 0 & 1 \end{array}\right), \qquad R_1 \to \frac{1}{3} R_1$$

$$\left(\begin{array}{cc|cc} 1 & \frac{1}{3} & \frac{1}{3} & 0 \\ 4 & 1 & 0 & 1 \end{array}\right), \qquad R_2 \to R_2 - 4R_1$$

$$\left(\begin{array}{cc|cc} 1 & \frac{1}{3} & \frac{1}{3} & 0 \\ 4-4(1) & 1-4(\frac{1}{3}) & 0-4(\frac{1}{3}) & 1-4(0) \end{array}\right),$$

$$\left(\begin{array}{cc|cc} 1 & \frac{1}{3} & \frac{1}{3} & 0 \\ 0 & \frac{-1}{3} & \frac{-4}{3} & 1 \end{array}\right), \qquad R_2 \to \frac{1}{\frac{-1}{3}} R_2$$

$$\left(\begin{array}{cc|cc} 1 & \frac{1}{3} & \frac{1}{3} & 0 \\ 0 & 1 & 4 & -3 \end{array}\right), \qquad R_1 \to R_1 - \frac{1}{3} R_2$$

$$\begin{pmatrix} 1 & \frac{1}{3} & | & \frac{1}{3} & 0 \\ 0 & 1 & | & 4 & -3 \end{pmatrix}, \qquad R_1 \to R_1 - \frac{1}{3}R_2$$

$$\begin{pmatrix} 1 & \frac{1}{3} - \frac{1}{3} & | & \frac{1}{3} - \frac{1}{3}(4) & 0 - \frac{1}{3}(-3) \\ 0 & 1 & | & 4 & -3 \end{pmatrix},$$

$$\begin{pmatrix} 1 & 0 & | & -1 & 1 \\ 0 & 1 & | & 4 & -3 \end{pmatrix}$$

Thus,

$$A^{-1} = \begin{pmatrix} -1 & 1 \\ 4 & -3 \end{pmatrix}$$

To check our answer, we must show that $AA^{-1} = I_n$ and $A^{-1}A = I_n$. We have that $AA^{-1} =$

$$\begin{pmatrix} -1 & 1 \\ 4 & -3 \end{pmatrix}$$

$$\begin{pmatrix} 3 & 1 \\ 4 & 1 \end{pmatrix} \begin{pmatrix} 3(-1) + 1(4) & 3(1) + 1(-1) \\ 4(-1) + 1(4) & 4(1) + 1(-3) \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = I_2$$

To check our answer, we must show that $AA^{-1} = I_n$ and $A^{-1}A = I_n$. We have that $A^{-1}A =$

$$\begin{array}{cc} & \begin{pmatrix} 3 & 1 \\ 4 & 1 \end{pmatrix} \\ \begin{pmatrix} -1 & 1 \\ 4 & -3 \end{pmatrix} & \begin{pmatrix} -1(3)+1(4) & -1(1)+1(1) \\ 4(3)+-3(4) & 4(1)+-3(1) \end{pmatrix} \end{array}$$

$$= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = I_2$$

Example 3

Find A^{-1} for

$$A = \begin{pmatrix} 1 & 1 \\ 3 & 3 \end{pmatrix}$$

Solution:

$$\left(\begin{array}{cc|cc} 1 & 1 & 1 & 0 \\ 3 & 3 & 0 & 1 \end{array}\right), \qquad R_2 \to R_2 - 3R_1$$

$$\left(\begin{array}{cc|cc} 1 & 1 & 1 & 0 \\ 3-3(1) & 3-3(1) & 0-3(1) & 1-3(0) \end{array}\right),$$

$$\left(\begin{array}{cc|cc} 1 & 1 & 1 & 0 \\ 0 & 0 & -3 & 1 \end{array}\right),$$

Since we couldn't reduce the matrix above, then it has no inverse!

Exercise 4

Find the inverse of

$$A = \begin{pmatrix} 1 & 0 \\ a & 1 \end{pmatrix}, \qquad a \in (-\infty, \infty)$$

Example 5

Find A^{-1} for

$$A = \begin{pmatrix} 2 & 1 & 0 \\ 4 & -1 & 5 \\ 1 & -1 & 2 \end{pmatrix}$$

Solution:

$$\left(\begin{array}{ccc|ccc} 2 & 1 & 0 & 1 & 0 & 0 \\ 4 & -1 & 5 & 0 & 1 & 0 \\ 1 & -1 & 2 & 0 & 0 & 1 \end{array}\right), \qquad R_1 \leftrightarrow R_3$$

$$\left(\begin{array}{ccc|ccc} 1 & -1 & 2 & 0 & 0 & 1 \\ 4 & -1 & 5 & 0 & 1 & 0 \\ 2 & 1 & 0 & 1 & 0 & 0 \end{array}\right), \qquad R_2 \to R_2 - 4R_1, R_3 \to R_3 - 2R_1$$

$$\left(\begin{array}{ccc|ccc} 1 & -1 & 2 & 0 & 0 & 1 \\ 4-4(1) & -1-4(-1) & 5-4(2) & 0-4(0) & 1-4(0) & 0-4(1) \\ 2-2(1) & 1-2(-1) & 0-2(2) & 1-2(0) & 0-2(0) & 0-2(1) \end{array}\right),$$

$$\left(\begin{array}{ccc|ccc} 1 & -1 & 2 & 0 & 0 & 1 \\ 0 & 3 & -3 & 0 & 1 & -4 \\ 0 & 3 & -4 & 1 & 0 & -2 \end{array}\right), \qquad R_2 \to \frac{1}{3}R_2$$

$$\left(\begin{array}{ccc|ccc} 1 & -1 & 2 & 0 & 0 & 1 \\ 0 & 1 & -1 & 0 & \frac{1}{3} & \frac{-4}{3} \\ 0 & 3 & -4 & 1 & 0 & -2 \end{array}\right), \qquad R_3 \to R_3 - 3R_2, R_1 \to R_1 + R_2$$

$$\left(\begin{array}{ccc|ccc} 1+0 & -1+1 & 2+-1 & 0+0 & 0+\frac{1}{3} & 1+\frac{-4}{3} \\ 0 & 1 & -1 & 0 & \frac{1}{3} & \frac{-4}{3} \\ 0 & 3-3(1) & -4-3(-1) & 1-3(0) & 0-3(\frac{1}{3}) & -2-3(\frac{-4}{3}) \end{array}\right),$$

$$\left(\begin{array}{ccc|ccc} 1 & 0 & 1 & 0 & \frac{1}{3} & \frac{-1}{3} \\ 0 & 1 & -1 & 0 & \frac{1}{3} & \frac{-4}{3} \\ 0 & 0 & -1 & 1 & -1 & 2 \end{array}\right), \qquad R_3 \to -R_3$$

$$\left(\begin{array}{ccc|ccc} 1 & 0 & 1 & 0 & \frac{1}{3} & \frac{-1}{3} \\ 0 & 1 & -1 & 0 & \frac{1}{3} & \frac{-4}{3} \\ 0 & 0 & 1 & -1 & 1 & -2 \end{array}\right), \qquad R_2 \to R_2 + R_3, R_1 \to R_1 - R_3$$

$$\left(\begin{array}{ccc|ccc} 1-0 & 0-0 & 1-1 & 0-(-1) & \frac{1}{3}-1 & \frac{-1}{3}-(-2) \\ 0+0 & 1+0 & -1+1 & 0+(-1) & \frac{1}{3}+1 & \frac{-4}{3}+(-2) \\ 0 & 0 & 1 & -1 & 1 & -2 \end{array}\right),$$

$$\left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & \frac{-2}{3} & \frac{5}{3} \\ 0 & 1 & 0 & -1 & \frac{4}{3} & \frac{-10}{3} \\ 0 & 0 & 1 & -1 & 1 & -2 \end{array}\right),$$

Thus,

$$A^{-1} = \begin{pmatrix} 1 & \frac{-2}{3} & \frac{5}{3} \\ -1 & \frac{4}{3} & \frac{-10}{3} \\ -1 & 1 & -2 \end{pmatrix}$$

To check our answer, we must show that $AA^{-1} = I_n$ and $A^{-1}A = I_n$. We have that $AA^{-1} =$

$$\begin{pmatrix} 1 & \frac{-2}{3} & \frac{5}{3} \\ -1 & \frac{4}{3} & \frac{-10}{3} \\ -1 & 1 & -2 \end{pmatrix}$$

$$\begin{pmatrix} 2 & 1 & 0 \\ 4 & -1 & 5 \\ 1 & -1 & 2 \end{pmatrix} \begin{pmatrix} 2(1)+1(-1)+0(-1) & 2(\frac{-2}{3})+1(\frac{3}{4})+0(1) & 2(\\ 4(1)+(-1)(-1)+5(-1) & 4(\frac{-2}{3})+(-1)(\frac{4}{3})+5(1) & 4(\frac{5}{3} \\ 1(1)+(-1)(-1)+2(-1) & 1(\frac{-2}{3})+(-1)(\frac{4}{3})+2(1) & 1(\frac{5}{3} \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = I_3$$

To check our answer, we must show that $AA^{-1} = I_n$ and $A^{-1}A = I_n$. We have that $A^{-1}A =$

$$\begin{pmatrix} 1 & \frac{-2}{3} & \frac{5}{3} \\ -1 & \frac{4}{3} & \frac{-10}{3} \\ -1 & 1 & -2 \end{pmatrix} \begin{pmatrix} 2 & 1 & 0 \\ 4 & -1 & 5 \\ 1 & -1 & 2 \end{pmatrix} \qquad \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = I_3$$

Properties

1. The inverse is unique. We already have seen this!
2. (Socks-shoes property) $(AB)^{-1} = B^{-1}A^{-1}$.
3. $(A^{-1})^{-1} = I_n$. Why?
4. $A^n = AA \dots A$ and $A^{-n} = A^{-1}A^{-1} \dots A^{-n}$.
5. $(A^T)^{-1} = (A^{-1})^T$