Section 4.6
Change of Basis

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MATHS 211: Linear Algebra

Goal:

1. Relation between bases.
2. Transition matrix.

Relation between bases

Example 1

Let $V = \mathbb{R}^2$. and let $\mathcal{B} = \left\{ \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \end{pmatrix} \right\}$ and $\mathcal{B}' = \left\{ \begin{pmatrix} 2 \\ -1 \end{pmatrix}, \begin{pmatrix} -1 \\ 1 \end{pmatrix} \right\}$ be two bases for V.

find

1. $\begin{pmatrix} 2 \\ 2 \end{pmatrix}_{\mathcal{B}}$.
2. $\begin{pmatrix} 2 \\ 2 \end{pmatrix}_{\mathcal{B}'}$.
3. Find v, w such that $(v)_{\mathcal{B}} = (5, 6)$ and $(w)_{\mathcal{B}'} = (5, 6)$
4. Find v such that $(v)_{\mathcal{B}} = (-2, 2)$ and then find $(v)_{\mathcal{B}'}$.

Transition Matrix

pause

Theorem 2

Let V be a vector space with two bases $\mathcal{B}$ (old basis) and $\mathcal{B}'$ (new basis). There is a matrix P such that for any v in V we have

$$(v)_{\mathcal{B}'} = P \cdot (v)_{\mathcal{B}}$$

We will often denote P by $P_{\mathcal{B} \to \mathcal{B}'}$ (**transition matrix**).
Question: How to find the transition matrix? Method 1:

$$[\text{ new basis } | \text{ old basis }] \to [I_n | \text{ transition matrix from old to new}]$$

Method 2: The column of the transition matrix are the coordinate vectors of the old basis relative to the new basis.

Example 3

Let $V = \mathbb{R}^2$. and let $\mathcal{B} = \left\{ \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \end{pmatrix} \right\}$ and $\mathcal{B}' = \left\{ \begin{pmatrix} 2 \\ -1 \end{pmatrix}, \begin{pmatrix} -1 \\ 1 \end{pmatrix} \right\}$ be two bases for V.

find

1. Find the transition matrix from $\mathcal{B}$ to $\mathcal{B}'$.
2. Given $(v)_{\mathcal{B}} = (-1, -3)$. Find $(v)_{\mathcal{B}'}$?

Example 4

Let $V = \mathcal{P}_2$. and let $\mathcal{B} = \{1, X, X^2\}$ and $\mathcal{B}' = \{1, 1 + X, 1 + X + X^2\}$ be two bases for V.

find

1. Find the transition matrix from $\mathcal{B}$ to $\mathcal{B}'$.
2. Given $p(X) = 4 - 2X + 6x^2$. Find $(p(X))_{\mathcal{B}'}$?

Example 5

Let $V = \mathcal{P}_2$. and let $\mathcal{B} = \{1, X, X^2\}$ and $\mathcal{B}' = \{1, 1+X, 1+X+X^2\}$ be two bases for V.

find

1. Find the transition matrix from $\mathcal{B}$ to $\mathcal{B}'$.
2. Given $p(X) = 4 - 2X + 6x^2$. Find $(p(X))_{\mathcal{B}'}$?

Inverse of a transition matrix

Question: What is the inverse of a transition matrix? What is $P^{-1}_{\mathcal{B}\to\mathcal{B}'}$?

Example 6

Let $V = \mathbb{R}^2$. and let $\mathcal{B} = \left\{ \begin{pmatrix} 2 \\ 2 \end{pmatrix}, \begin{pmatrix} 1 \\ -2 \end{pmatrix} \right\}$ and $\mathcal{B}' = \left\{ \begin{pmatrix} -1 \\ 2 \end{pmatrix}, \begin{pmatrix} 3 \\ 0 \end{pmatrix} \right\}$ be two bases for V.

find

1. Find the transition matrix from $\mathcal{B}$ to $\mathcal{B}'$.
2. Find the transition matrix from $\mathcal{B}'$ to $\mathcal{B}$.
3. Multiple the two matrices together.

Theorem 7

$$P_{\mathcal{B}'\to\mathcal{B}} = P^{-1}_{\mathcal{B}\to\mathcal{B}'}$$

Example 8

Let V be the space spanned by $f_1 = \cos x$, $f_2 = \sin x$.

find

1. Show that $g_1 = 2\sin x + \cos x$ and $g_2 = 3\cos x$ form a basis for V.
2. Find the transition matrix from $\mathcal{B} = \{g_1, g_2\}$ to $\mathcal{B}' = \{f_1, f_2\}$.
3. Given $h = 2\sin x - 5\cos x$. Find $(h)_{\mathcal{B}}$ and use it to find $(h)_{\mathcal{B}'}$.