Section 5.1
Eigenvalues

Dr. Abdulla Eid

College of Science

MATHS 211: Linear Algebra

Goal:

1. Define the **Eigenvalues** of a matrix.
2. The **characteristic polynomial** and the Eigenvalues of a matrix.
3. Define and find basis for the **Eigenvectors** of a matrix.

Definition 1

If A is an $n \times n$ matrix, then a **nonzero** vector $\mathbf{x}$ in $\mathbb{R}^n$ is called an **Eigenvector** of A if

$$A\mathbf{x} = \lambda \mathbf{x}$$

for some scalar $\lambda \in \mathbb{R}$. The scalar λ is an **Eigenvalue** of A and $\mathbf{x}$ is said to be the **Eigenvector** corresponding to λ.

Characteristic Polynomial of a matrix

Theorem 2

If A is an $n \times n$ matrix, then λ is an Eigenvalue if and only if

$$\det(\lambda I_n - A) = \mathbf{0}$$

This is called the characteristic polynomial of A.

Example 3

Find the Eigenvalues of

$$A = \begin{pmatrix} 2 & -1 \\ 10 & -9 \end{pmatrix}$$

Example 4

Find the Eigenvalues of

$$A = \begin{pmatrix} 3 & 0 \\ 5 & 3 \end{pmatrix}$$

Example 5

Find the Eigenvalues of

$$A = \begin{pmatrix} -2 & -7 \\ 1 & 2 \end{pmatrix}$$

Example 6

Find the Eigenvalues of

$$A = \begin{pmatrix} 5 & 1 & 3 \\ 0 & -1 & 0 \\ 0 & 1 & 2 \end{pmatrix}$$

Example 7

Find the Eigenvalues of

$$A = \begin{pmatrix} 0 & 0 & 2 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & -2 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Number of Eigenvalues

The characteristic polynomial of any $n \times n$ matrix is of the form

$$p(\lambda) = \lambda^n + c_{n-1}\lambda^{n-2} + \cdots + c_2\lambda^2 + c_1\lambda + c_0$$

So we get at most n **real** Eigenvalues and exactly n complex Eigenvalues.

Example 8

Find the Eigenvalues of

$$A = \begin{pmatrix} 5 & 7590 & 2 & -2001 \\ 0 & 7 & 1020 & 1010 \\ 0 & 0 & -2 & 230 \\ 0 & 0 & 0 & 99 \end{pmatrix}$$

Theorem 9

If A is $n \times n$ **triangular** *matrix (lower, upper, diagonal), then the Eigenvalues of A are the entries on the main diagonal.*

Theorem 10

If A is $n \times n$ matrix with Eigenvalue λ, then A^k has λ^k as Eigenvalue with the same Eigenvector.

Example 11

Find the Eigenvalues of A^{25} of

$$A = \begin{pmatrix} -1 & -2 & -2 \\ 1 & 2 & 1 \\ -1 & -1 & 0 \end{pmatrix}$$

Theorem 12

A square matrix A is invertible if and only if $\lambda = 0$ is **not** *an Eigenvalue.*

Example 13

Check all the matrices in the previous examples and determine which of the ones is invertible.

Note:

$$\det(A) = \frac{c_0}{(-1)^n}$$

Example 14

Find $\det(A)$ given that A has the characteristic polynomial $p(\lambda)$.

(1) $p(\lambda) = \lambda^3 + 2\lambda^2 - 4\lambda - 5$.

(2) $p(\lambda) = \lambda^5 + 3\lambda^2 - 2\lambda + 12$.

Theorem 15

If λ is an Eigenvalue of an invertible matrix A, then $\frac{1}{\lambda}$ is an Eigenvalue for A^{-1}.

Example 16

Find bases for the Eigenspace of

$$A = \begin{pmatrix} 2 & -1 \\ 10 & -9 \end{pmatrix}$$

Example 17

Find bases for the Eigenspace of

$$A = \begin{pmatrix} 2 & 0 \\ 5 & 2 \end{pmatrix}$$

Example 18

Find the Eigenspace of

$$A = \begin{pmatrix} -2 & 0 & 1 \\ -6 & -2 & 0 \\ 19 & 5 & -4 \end{pmatrix}$$