

Section 6.1

Inner Product

Dr. Abdulla Eid

College of Science

MATHS 211: Linear Algebra

Goal:

- 1 Define Inner Product.
- 2 Examples of the inner product.
- 3 Properties of the inner product.

Dr. Abdulla Eid

Definition 1

General Inner Product An **inner product** is a function $\langle \cdot, \cdot \rangle : V \times V \rightarrow \mathbb{R}$ such that

- 1 $\langle \mathbf{u}, \mathbf{v} \rangle = \langle \mathbf{v}, \mathbf{u} \rangle$. (Symmetric axiom)
- 2 $\langle \mathbf{u} + \mathbf{v}, \mathbf{w} \rangle = \langle \mathbf{u}, \mathbf{w} \rangle + \langle \mathbf{v}, \mathbf{w} \rangle$. (Additive axiom)
- 3 $\langle k\mathbf{u}, \mathbf{w} \rangle = k \langle \mathbf{u}, \mathbf{w} \rangle$. (homogeneity axiom)
- 4 $\langle \mathbf{u}, \mathbf{u} \rangle \geq 0$ and $\langle \mathbf{u}, \mathbf{u} \rangle = 0$ if and only if $\mathbf{u} = \mathbf{0}$ (positive definiteness axiom)

Example 2

Let $V = \mathbb{R}^2$. Define

$$\langle \mathbf{u}, \mathbf{v} \rangle = u_1v_1 + u_2v_2$$

Example 3

Find $\left\langle \begin{pmatrix} 3 \\ 4 \end{pmatrix}, \begin{pmatrix} -2 \\ 5 \end{pmatrix} \right\rangle$,

$$\left\langle \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right\rangle,$$

$$\left\langle \begin{pmatrix} -1 \\ 2 \end{pmatrix}, \begin{pmatrix} 1 \\ -2 \end{pmatrix} \right\rangle$$

Example 4

Let $V = \mathbb{R}^2$. Define

$$\langle \mathbf{u}, \mathbf{v} \rangle = u_1 v_1 + u_2 v_2 + \cdots + u_n v_n$$

This is called the **dot** or **Euclidean** product.

Example 5

Find $\left\langle \begin{pmatrix} -5 \\ 3 \\ 4 \end{pmatrix}, \begin{pmatrix} 11 \\ -2 \\ 5 \end{pmatrix} \right\rangle$,

$$\left\langle \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ -2 \\ 1 \end{pmatrix} \right\rangle,$$

$$\left\langle \begin{pmatrix} -1 \\ 2 \\ -1 \end{pmatrix}, \begin{pmatrix} 2 \\ -2 \\ 1 \end{pmatrix} - 2 \right\rangle$$

Example 6

Let $V = \mathbb{R}^2$. Define

$$\langle \mathbf{u}, \mathbf{v} \rangle = 3u_1v_1 + 5u_2v_2$$

Example 7

Find $\left\langle \begin{pmatrix} 3 \\ 4 \end{pmatrix}, \begin{pmatrix} -2 \\ 5 \end{pmatrix} \right\rangle$,

$$\left\langle \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right\rangle,$$

$$\left\langle \begin{pmatrix} -1 \\ 2 \end{pmatrix}, \begin{pmatrix} 1 \\ -2 \end{pmatrix} \right\rangle$$

Weighted inner product \mathbb{R}^n

Example 8

Let $V = \mathbb{R}^n$ and let w_1, \dots, w_n be positive real numbers (**weights**).

Define

$$\langle \mathbf{u}, \mathbf{v} \rangle = w_1 u_1 v_1 + w_2 u_2 v_2 + \cdots + w_n u_n v_n$$

This is called the **dot** or **Euclidean** product.

Norm and distance

Definition 9

Let V be a real inner space. The **norm** (or **length**) of a vector \mathbf{v} of V is denoted by $\|\mathbf{v}\|$ and is defined by

$$\|\mathbf{v}\| = \sqrt{\langle \mathbf{v}, \mathbf{v} \rangle}$$

and the distance between any two vectors \mathbf{u} and \mathbf{v} is defined by

$$d(\mathbf{u}, \mathbf{v}) = \|\mathbf{u} - \mathbf{v}\| = \sqrt{\langle \mathbf{u} - \mathbf{v}, \mathbf{u} - \mathbf{v} \rangle}$$

A vector of norm 1 is called a **unit vector**.

Example 10

Let $V = \mathbb{R}^2$ with the Euclidean product. Let $\mathbf{u} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$, $\mathbf{v} = \begin{pmatrix} 3 \\ 2 \end{pmatrix}$,

$\mathbf{w} = \begin{pmatrix} -1 \\ 0 \end{pmatrix}$, and $k = 5$. Find the following:

- (a) $\langle \mathbf{u}, \mathbf{w} \rangle$ (b) $\langle k\mathbf{u}, \mathbf{v} \rangle$
(c) $\langle \mathbf{u} + \mathbf{v}, \mathbf{w} \rangle$ (d) $\|\mathbf{u}\|$
(e) $d(\mathbf{u}, \mathbf{v})$ (f) $\|\mathbf{u} - k\mathbf{v}\|$

Example 11

Let $V = \mathbb{R}^2$. Define

$$\langle \mathbf{u}, \mathbf{v} \rangle = 3u_1v_1 + 5u_2v_2$$

Let $\mathbf{u} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$, $\mathbf{v} = \begin{pmatrix} 3 \\ 2 \end{pmatrix}$, $\mathbf{w} = \begin{pmatrix} -1 \\ 0 \end{pmatrix}$, and $k = 5$. Find the following:

- (a) $\langle \mathbf{u}, \mathbf{w} \rangle$ (b) $\langle k\mathbf{u}, \mathbf{v} \rangle$
(c) $\langle \mathbf{u} + \mathbf{v}, \mathbf{w} \rangle$ (d) $\|\mathbf{u}\|$
(e) $d(\mathbf{u}, \mathbf{v})$ (f) $\|\mathbf{u} - k\mathbf{v}\|$

Matrix inner product on \mathbb{R}^n

Example 12

Let $V = \mathbb{R}^n$ and let A be invertible $n \times n$ matrix. Define

$$\langle \mathbf{u}, \mathbf{v} \rangle = \mathbf{A}\mathbf{u} \cdot \mathbf{A}\mathbf{v}$$

This is called the **inner product on \mathbb{R}^n generated by A**

The Euclidean inner product is a special case with $A = I_n$ and the weighted inner product is a special case with

$$A = \begin{pmatrix} \sqrt{w_1} & 0 & 0 & \dots & 0 \\ 0 & \sqrt{w_2} & 0 & \dots & 0 \\ \cdot & \cdot & 0 & \cdot & \cdot \\ 0 & 0 & 0 & \dots & \sqrt{w_n} \end{pmatrix}$$

Example 13

Let $V = \mathbb{R}^2$ and $A = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$. Define

$$\langle \mathbf{u}, \mathbf{v} \rangle = A\mathbf{u} \cdot A\mathbf{v}$$

Let $\mathbf{u} = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$, $\mathbf{v} = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$, and $\mathbf{w} = \begin{pmatrix} 0 \\ -1 \end{pmatrix}$. Find the following:

- (a) $\langle \mathbf{u}, \mathbf{w} \rangle$ (b) $\langle \mathbf{u}, \mathbf{v} \rangle$
(c) $\|\mathbf{u}\|$ (d) $\|\mathbf{u}\|$
(e) $d(\mathbf{u}, \mathbf{v})$ (f) $\|\mathbf{u} - \mathbf{v}\|^2$

Inner product on square matrices

Definition 14

Let $V = \text{Mat}(n, n, \mathbb{R})$ and let U, V be invertible $n \times n$ matrices. Define

$$\langle U, V \rangle = \text{tr}(U^T V)$$

if U, V are two by two matrices, then what is $\langle U, V \rangle$?

Inner product on matrices

Example 15

Find $\langle U, V \rangle$ for

$$\textcircled{1} \quad U = \begin{pmatrix} 3 & -2 \\ 4 & 8 \end{pmatrix}, V = \begin{pmatrix} -1 & 3 \\ 1 & 1 \end{pmatrix}$$

$$\textcircled{2} \quad U = \begin{pmatrix} 1 & 2 \\ -3 & 5 \end{pmatrix}, V = \begin{pmatrix} 4 & 6 \\ 0 & 8 \end{pmatrix}$$

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Inner product on Polynomials

Definition 16

Let $V = \mathbb{P}_n$. Let $p = a_0 + a_1X + a_2X^2 + \cdots + a_nX^n$ and $q = b_0 + b_1X + b_2X^2 + \cdots + b_nX^n$ be two polynomials. Define the **standard inner product** to be

$$\langle p, q \rangle = a_0b_0 + a_1b_1 + a_2b_2 + \cdots + a_nb_n$$

If x_0, x_1, \dots, x_n are distinct real numbers, define the **evaluation inner product** to be

$$\langle p, q \rangle = p(x_0)q(x_0) + p(x_1)q(x_1) + p(x_2)q(x_2) + \cdots + p(x_n)q(x_n)$$

Inner product on matrices

Example 17

Find $\langle p, q \rangle$, $\|p\|$, $\|q\|^2$, and $d(p, q)$ for

- 1 $p = 3 - X + 2X^2, q = 2 - 4X^2$.
- 2 $p = X + 3X^2, q = 2 - X + 4X^2$.
- 3 $p = 1 - 2X + 3X^2, q = 4 + X^2$ with $x_0 = 2, x_1 = -1, x_2 = 1$.

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Inner Product on $C([a, b])$

Example 18

Let $V = C([a, b])$ be the vector space of continuous functions on an interval $[a, b]$. Define

$$\langle f, g \rangle = \int_a^b f(x)g(x) dx$$

Let $f(x) = x + 1$, $g(x) = x - 1$, $h(x) = 5$, and $a = 1, b = 2$. Find the following:

- (a) $\langle \mathbf{u}, \mathbf{w} \rangle$ (b) $\langle 8\mathbf{u}, \mathbf{v} \rangle$
(c) $\langle u + v, w \rangle$ (d) $\|\mathbf{u}\|$

Non-example, Lorentian Inner product

Example 19

Let $V = \mathbb{R}^4$. Define

$$\langle \mathbf{u}, \mathbf{v} \rangle = x_1x_2 + y_1y_2 + z_1z_2 - t_1t_2$$

This is called the **Lorentzian** inner product. This is of central importance in Einstein's theory of special relativity.

Note: This is not an inner product! Why?

Example 20

Find $\left\langle \begin{pmatrix} -5 \\ 3 \\ 4 \\ 3 \end{pmatrix}, \begin{pmatrix} 11 \\ -2 \\ 5 \\ 2 \end{pmatrix} \right\rangle, \left\langle \begin{pmatrix} 1 \\ 0 \\ 1 \\ 7 \end{pmatrix}, \begin{pmatrix} 0 \\ -2 \\ 1 \\ 2 \end{pmatrix} \right\rangle$.

Continue

Two space time vectors $X_1 = \begin{pmatrix} x_1 \\ y_1 \\ z_1 \\ t_1 \end{pmatrix}$ and $X_2 = \begin{pmatrix} x_2 \\ y_2 \\ z_2 \\ t_2 \end{pmatrix}$ are

Separated by a distance $\sqrt{\langle X_1, X_2 \rangle}$ if $\langle X_1, X_2 \rangle \geq 0$

Separated by a time $\sqrt{-\langle X_1, X_2 \rangle}$ if $\langle X_1, X_2 \rangle \leq 0$

Theorem 21

Norm and Distance Let V be an inner product vector space. Then,

- 1 $\langle \mathbf{0}, \mathbf{v} \rangle = 0$.
- 2 $\|\mathbf{v}\| \geq 0$ with equality if and only if $\mathbf{v} = \mathbf{0}$.
- 3 $\|k\mathbf{v}\| = |k|\|\mathbf{v}\|$.
- 4 $d(\mathbf{u}, \mathbf{v}) = d(\mathbf{v}, \mathbf{u})$.

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Example 22

Prove that in any inner product vector space, we have

$$\|\mathbf{u} + \mathbf{v}\|^2 + \|\mathbf{u} - \mathbf{v}\|^2 = 2\|\mathbf{u}\|^2 + 2\|\mathbf{v}\|^2$$

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Example 23

Prove that in any inner product vector space, we have

$$\langle \mathbf{u}, \mathbf{v} \rangle = \frac{1}{4} \|\mathbf{u} + \mathbf{v}\|^2 - \frac{1}{4} \|\mathbf{u} - \mathbf{v}\|^2 = 2\|\mathbf{u}\|^2 + 2\|\mathbf{v}\|^2$$

Dr. Abdulla E.