Section 8.1
Linear Transformation
Part 1: Definition and Examples

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MATHS 211: Linear Algebra

Goal:

1. Define Linear Transformations.
2. Examples of Linear Transformations.
3. Finding the linear transformation from a basis.
4. Kernel and Range of a linear transformations.
5. Properties and of Kernel and Images.
6. Rank and Nullity of a linear transformation.

Definition 1

A **linear transformation** is a function $T : V \to W$ with

1. $T(k\mathbf{v}) = kT(\mathbf{v})$. (Homogeneity Property)
2. $T(\mathbf{u}+\mathbf{v}) = T(\mathbf{u}) + T(\mathbf{v})$. (Additive Property)

In the special case, where $V = W$, the linear transformation T is called **linear operator**.

Example 2

Define a mapping $T : \mathbb{R}^2 \to \mathbb{R}^2$ by

$$T\left(\begin{pmatrix} x \\ y \end{pmatrix}\right) = \left(\begin{pmatrix} 2x - y \\ x + 3y \end{pmatrix}\right)$$

(a) Find the image of e_1, e_2 under T.
(b) Give a description of all vectors in $\mathbb{R}^2$ that are mapped to the zero vector.
(c) Show that the mapping is a linear operator.

Simple Consequences from the definition

- $$T(k_1\mathbf{v}_1 + k_2\mathbf{v}_2) = T(k_1\mathbf{v}_1) + T(k_2\mathbf{v}_2)$$
  $$= k_1T(\mathbf{v}_1) + k_2T(\mathbf{v}_2)$$
- In general,
  $$T(k_1\mathbf{v}_1 + k_2\mathbf{v}_2 + \cdots + k_n\mathbf{v}_n) = k_1T(\mathbf{v}_1) + k_2T(\mathbf{v}_2) + \cdots + k_nT(\mathbf{v}_n)$$
- $$T(\mathbf{0}) = T(\mathbf{0} + \mathbf{0})$$
  $$T(\mathbf{0}) = T(\mathbf{0}) + T(\mathbf{0})$$
  $$\mathbf{0}_W = T(\mathbf{0})$$ Zero goes to zero
- $$T(\mathbf{u} - \mathbf{v}) = T(\mathbf{u}) - T(\mathbf{v})$$

Matrix Transformation

Example 3

Let A be any $m \times n$ matrix. Define the linear transformation $T_A : \mathbb{R}^n \to \mathbb{R}^m$ by

$$T_A(\mathbf{x}) = A\mathbf{x}$$

Example 4

Let $A = \begin{pmatrix} 1 & 2 & -1 \\ -1 & 3 & 2 \end{pmatrix}$. Find the image of the matrix transformation of

(a) $\mathbf{u} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$ (b) $\mathbf{v} = \begin{pmatrix} 7 \\ -1 \\ 5 \end{pmatrix}$ (c) $\mathbf{u} + \mathbf{v}$

Zero Transformation

Example 5

Let V be any vector space. Define the linear transformation $T : V \to W$ by

$$T(\mathbf{x}) = \mathbf{0}_W$$

Identity Operator

Example 6

Let V be any vector space. Define the linear transformation (operator) $T : V \to V$ by

$$T(\mathbf{x}) = \mathbf{x}$$

Dilation and Contraction Operators

Example 7

Let V be any vector space. Define the linear operator $T_k : V \to V$ by

$$T_k(\mathbf{x}) = k\mathbf{x}$$

If $0 < k < 1$, then T_k is called the **contraction** of V with factor k and if $k > 1$, it is called the **dilation** of V with factor k.

A Linear Transformation from $\mathbb{P}_n$ to $\mathbb{P}_{n+1}$

Example 8

Define the linear operator $T : \mathbb{P}_n \to \mathbb{P}_n$ by

$$T(p(X)) = XP(X)$$

$$T(c_0 + c_1X + c_2X^2 + \cdots + c_nX^n) = c_0X + c_1X^2 + c_2X^3 + \cdots + c_nX^{n+1}$$

Example 9

Define a mapping $T : \mathbb{P}_2 \to \mathbb{P}_2$ by

$$T\left(p(X)\right) = p''(X) - 2p'(X) + p(X)$$

(a) Find the image of $\mathbf{u} = X^2 - 3X + 1, \mathbf{v} = -X - 1, \mathbf{u} + \mathbf{v}$ under T.
(b) Is the mapping T a linear operator.

Linear Transformation of the Matrices

Example 10

Define the linear transformation $T : \text{Mat}(m, n, \mathbb{R}) \to \text{Mat}(n, m, \mathbb{R})$ by

$$T(A) = A^T$$

Linear Transformation of the Matrices

Example 11

Define the function $T : \text{Mat}(m, n, \mathbb{R}) \to \mathbb{R}$ by

$$T(A) = \det(A)$$

Is T a linear transformation?

Evaluation Transformation

Example 12

Let $c_1, c_2, \ldots, c_n$ be real numbers. Define the linear transformation $T : \text{Maps}(\mathbb{R}, \mathbb{R}) \to \mathbb{R}^n$ by

$$T(f) = (f(c_1), f(c_2), \ldots, f(c_n))$$

Example 13

Define a mapping $T : \mathbb{R}^2 \to \mathbb{R}^3$ by

$$T\left(\begin{pmatrix} x \\ y \end{pmatrix}\right) = \left(\begin{pmatrix} \cos x \\ \sin x \\ \sin y \end{pmatrix}\right)$$

(a) Show that the mapping is a linear operator.

2- Finding Linear Transformation from the image of a basis

Example 14

Consider the basis $\mathcal{B} = \{\mathbf{v}_1, \mathbf{v}_2\}$ of $\mathbb{R}^2$, where

$$\mathbf{v}_1 = \begin{pmatrix} 2 \\ 1 \end{pmatrix}, \quad \mathbf{v}_2 = \begin{pmatrix} -1 \\ 3 \end{pmatrix}$$

and let $T : \mathbb{R}^2 \to \mathbb{R}^3$ be the linear transformation such that

$$T(\mathbf{v}_1) = \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix}, \quad T(\mathbf{v}_2) = \begin{pmatrix} 0 \\ 3 \\ 5 \end{pmatrix}$$

Find a formula for $T(x_1, x_2)$ in general and use it to find $T(2, -3)$ and $T(4, -1)$.

Finding Linear Transformation from the image of a basis

Example 15

Consider the basis $\mathcal{B} = \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ of $\mathbb{R}^3$, where

$$\mathbf{v}_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \quad \mathbf{v}_2 = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \quad \mathbf{v}_3 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

and let $T : \mathbb{R}^3 \to \mathbb{R}^3$ be the linear transformation such that

$$T(\mathbf{v}_1) = \begin{pmatrix} -1 \\ 2 \\ 4 \end{pmatrix}, \quad T(\mathbf{v}_2) = \begin{pmatrix} 0 \\ 3 \\ 2 \end{pmatrix}, \quad T(\mathbf{v}_3) = \begin{pmatrix} 1 \\ 5 \\ -1 \end{pmatrix}.$$

Find a formula for $T(x_1, x_2, x_3)$ in general and use it to find $T(2, 4, -1)$ and $T(4, 3, 4)$.

Definition 16

If $T : V \to W$ be a linear transformation. The set of all vectors in V that T maps into zero is called the **kernel** of T and denoted by $\ker(T)$.the set of all vectors in W that are images under T of at least one vector in V is called the **range** of T and is denoted by $R(T)$.

$$\ker(T) := \{\mathbf{v} \in V \mid T(\mathbf{v}) = \mathbf{0}_W\}$$

$$R(T) := \{\mathbf{w} \in W \mid T(\mathbf{v}) = \mathbf{w},\ \text{for some } \mathbf{v} \in V\}$$

Matrix Transformation

Example 17

Let A be any $m \times n$ matrix. Define the linear transformation $T_A : \mathbb{R}^n \to \mathbb{R}^m$ by

$$T_A(\mathbf{x}) = A\mathbf{x}$$

$Ker(T_A)$ is the Null space of A while $R(T_A)$ is the Column space of A

Zero Transformation

Example 18

Let V be any vector space. Define the linear transformation $T : V \to W$ by

$$T(\mathbf{x}) = \mathbf{0}_W$$

Identity Operator

Example 19

Let V be any vector space. Define the linear transformation (operator) $T : V \to V$ by

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Dilation and Contraction Operators

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$$T(c_0 + c_1X + c_2X^2 + \cdots + c_nX^n) = c_0X + c_1X^2 + c_2X^3 + \cdots + c_nX^{n+1}$$

Theorem 22

Let $T : V \to W$ be a linear transformation. $\ker(T)$ *and* $R(A)$ *are both subspaces of* V *and* W.

1. Dimension of the $\ker(T)$ is called the **nullity** of T.
2. Dimension of the $R(A)$ is called the **rank** of T.

Theorem 23

Dimension Theorem Let $T : V \to W$ be a linear transformation, then

$$rank(T) + Nullity(T) = \dim(V) = n$$

Example 24

Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be the linear transformation given by

$$T(x, y) = (x - 3y, -2x + 6y)$$

Which of the following vectors are in $R(T)$?
$(1, -2)$, $(3, 1)$, $(-2, 4)$ Which of the following vectors are in $ker(T)$?
$(1, -3)$, $(3, 1)$, $(-6, -2)$ Find a basis for the $\ker(T)$? Find a basis for the $R(T)$? Verify the formula in the dimension theorem?

Example 25

Let $T : \mathbb{P}_2 \to \mathbb{P}_3$ be the linear transformation given by

$$T(p(X)) = (X+1)p(X)$$

Which of the following vectors are in $R(T)$?
$X + X^2$, $1 + X$, $3 - X^2$ Which of the following vectors are in $\ker(T)$?
X^2, 0, $X + 1$ Find a basis for the $\ker(T)$? Find a basis for the $R(T)$?
Verify the formula in the dimension theorem?