§10.2 - Sequences

Definition:

A sequence is a list of real numbers indexed by the natural numbers

$$(a_1 , a_2 , a_3 , a_4 , \ldots , a_n , \ldots)$$

Each element of the sequence is called a term.

a_1 is the first term

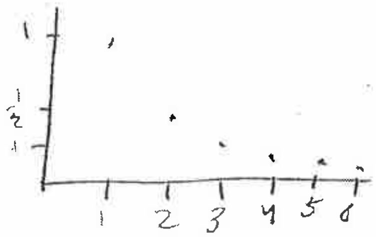
a_2 is the second term

$\vdots$

a_n is the n^{th} term

Example:

a. $a_n = \frac{1}{n} \to (\overset{a_1}{1} , \overset{a_2}{\frac{1}{2}} , \overset{a_3}{\frac{1}{3}} , \ldots , \frac{1}{n}) \ldots$



b. $b_n = \left(\frac{5+n}{n} \right)^n \to \left(6 , \left(\frac{7}{2}\right)^2 , \left(\frac{8}{3}\right)^2 , \ldots \right)$

1 - Limits

The limit of a sequence is the usual notion of a limit

$$\lim_{n \to \infty} a_n = L$$

and if L is a real number, we say that the sequence converges. Otherwise, we call it diverges.

Example 2: Find the limit of

(a) $a_n = \frac{1}{n}$

(b) $a_n = \frac{3n^3}{n^3+1}$

(c) $b_n = \left(\frac{5+n}{n}\right)^n$ --- recall $e^x = \lim_{n\to\infty}\left(1+\frac{x}{n}\right)^n$

(d) $c_n = n^{\frac{1}{n}}$.

Solution:

(a) $\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{1}{n} = 0$ --- converges

(b) $\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{3n^3}{n^3+1} = \lim_{n\to\infty} \frac{3\cancel{n^3}}{\cancel{n^3}(1+\frac{1}{n^3})} = 3$

(c) $\lim_{n\to\infty} b_n = \lim_{n\to\infty}\left(\frac{5+n}{n}\right)^n = \lim_{n\to\infty}\left(\frac{5}{n}+\frac{n}{n}\right)^n = \lim_{n\to\infty}\left(1+\frac{5}{n}\right)^n = e^5$

d) $\lim_{n\to\infty} n^{\frac{1}{n}} = L \rightsquigarrow \lim_{n\to\infty} \ln n^{\frac{1}{n}} = \ln L \rightsquigarrow \lim_{n\to\infty} \frac{\ln n}{n} = \ln L$

Now we apply the L'Hopital rule, so $\lim_{n\to\infty} \frac{\frac{1}{n}}{1} = \ln L$

$\lim_{n\to\infty} \frac{1}{n} = \ln L \rightsquigarrow 0 = \ln L \sim L = e^0 = 1$

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2 - Monotonic Bounded Theorem

Definition:

A sequence $\{a_n\}$ is called increasing if $a_n \leq a_{n+1}$

A sequence $\{b_n\}$ is called decreasing if $a_n \geq a_{n+1}$

A sequence that is either increasing or decreasing is called a <u>monotone</u> sequence

A sequence is called <u>bounded</u> if there exist $M > 0$ such that

$$|a_n| \le M \quad , \forall n \in \mathbb{N}.$$

<u>Example 3 :-</u>

(i) $a_n = \frac{1}{n}$ is decreasing sequence. (check $a_n \ge a_{n+1}$ or find the derivative).

$$a_n \boxed{\ge} a_{n+1} \checkmark$$

$$\frac{1}{n} \boxed{\ge} \frac{1}{n+1}$$

$$n+1 \boxed{\ge} n$$

Moreover, it is <u>bounded</u> since $|a_n| = \frac{1}{n} < 1 \quad , \forall n \in \mathbb{N}$

(ii) $a_n = \frac{n}{n^2+1}$, (iii) $a_n = ne^{-n}$

<u>Example :-</u>

Suppose you have a hot cup of coffee and you put it in the desk to cool. Assume every minute you check the temperature and you record the as $\{T_1, T_2, T_3, \ldots\}$. Then we have

1- This sequence is decreasing (since the coffee is cooling)

2- The sequence is bounded (it cannot go less than the room temperature).

So ultimately, we expect the coffee in the long run to be as the same temperature as the room!

Theorem: (Bounded Monotonic Sequences)

A bounded monotonic sequence is convergent

Find the limit of each of the following

(i) $a_n = (-1)^n$ (ii) $\{0, 1, 0, 0, 1, 0, 0, 0, 1, \ldots\}$

(iii) $a_n = r^n$.

Discuss the following recursively defined sequence

- write the first few term
- Discuss increasing / decreasing / boundedness
- Find the limits.

(1) $a_1 = 2$, $a_{n+1} = \frac{1}{2}(a_n + 6)$

(2) $a_1 = 1$, $a_{n+1} = 3 - \frac{1}{a_n}$

(3) $\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{2+\cdots}}}}$, $a_1 = \sqrt{2}$, $a_{n+1} = \sqrt{2+a_n}$

$\Updownarrow$

(4) $\{\sqrt{2}, \sqrt{2\sqrt{2}}, \sqrt{2\sqrt{2\sqrt{2}}}, \ldots\}$