# § 4.5 - Indeterminate Forms and L'Hopital Rules

Motivational Example

Consider

1. $$\lim_{x \to 1} \frac{\ln x}{x-1} \xrightarrow{\text{direct}} \frac{0}{0}$$ ... "indeterminate form of type $\frac{0}{0}$"

2. $$\lim_{x \to \infty} \frac{x^2-1}{3x^2+7x+1} \xrightarrow{\text{direct}} \frac{\infty}{\infty}$$ "indeterminate form of type $\frac{\infty}{\infty}$"

Question: How to find these limits $\lim_{x \to a} \frac{f(x)}{g(x)}$ ?

L'Hopital's Rule:

Suppose $f$ and $g$ are differentiable and $g'(x) \neq 0$ on an open interval $I$ that contains $a$ (except possibly at $a$). Suppose

$$\lim_{x \to a} f(x) = 0 \quad \text{and} \quad \lim_{x \to a} g(x) = 0 \qquad \left(\frac{0}{0}\right)$$

or

$$\lim_{x \to \cancel{\pm\infty}} f(x) = \pm\infty \quad \text{and} \quad \lim_{x \to a} g(x) = \pm\infty \qquad \left(\frac{\infty}{\infty}\right)$$

Then,

$$\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f'(x)}{g'(x)} \quad \text{(if it exists)}$$

Example 1: Find $\lim_{x\to 1} \frac{\ln x}{x-1}$

Solution:

Since $\lim_{x\to 1} \ln x = \ln 1 = 0$ , $\lim_{x\to 1} x-1 = 0$, so we have $\frac{0}{0}$.

$$\lim_{x\to 1} \frac{\ln x}{x-1} = \lim_{x\to 1} \frac{(\ln x)'}{(x-1)'} = \lim_{x\to 1} \frac{\frac{1}{x}}{1} = \lim_{x\to 1} \frac{1}{x} = 1.$$

Exercise 1: Find $\lim_{x\to 0^+} \frac{\ln x}{x}$

Example 2: Find $\lim_{x\to \infty} \frac{e^x}{x}$

Solution:

$$\lim_{x\to \infty} \frac{e^x}{x} = \lim_{x\to \infty} \frac{e^x}{1} = \infty.$$

Exercise 2: Find $\lim_{x\to 0} \frac{\sin^{-1} x}{x}$

$$\lim_{x\to 0} \frac{\sin^{-1} x}{x} = \lim_{x\to 0} \frac{\frac{1}{\sqrt{1-x^2}}}{1} = \lim_{x\to 0} \frac{1}{\sqrt{1-x^2}} = 1$$

Example 3: $\lim_{x\to 0} \frac{x-\sin x}{x-\tan x}$

Example 4: $\lim_{x \to 0} \frac{x+\sin x}{x+\cos x} = \frac{0}{1} = 0$

Example 5: $\lim_{x \to \infty} \frac{(\ln x)^2}{x} = \frac{\infty}{\infty}$

$\overset{(H)}{=} \lim_{x \to \infty} = \frac{2\ln x}{x}$

$\overset{(H)}{=} \lim_{x \to \infty} = \frac{2}{x} = 0$

Exercise 3: Find $\lim_{x \to 1} \frac{x^a - 1}{x^b - 1} = \frac{0}{0}$

$\overset{(H)}{=} \frac{ax^{a-1}}{bx^{b-1}} = \frac{a}{b}$

Example 6: Find $\lim_{x \to 1} \frac{1 - x + \ln x}{1 + \cos \pi x} = \frac{0}{0}$

$\overset{(H)}{=} \lim_{x \to 1} \frac{-1 + \frac{1}{x}}{-\pi \sin \pi x} = \frac{0}{0}$

$\overset{(H)}{=} \lim_{x \to 1} = \frac{-\frac{1}{x^2}}{-\pi^2 \cos \pi x} = \frac{-1}{\pi^2}$

## 2- Indeterminate Products

$$\lim_{x\to a} f(x)\,g(x) = 0\cdot\infty \leftarrow \text{ indeterminate form of the form } 0\cdot\infty.$$

Idea: Convert $fg$ into either $\dfrac{f}{\frac{1}{g}}$ or $\dfrac{g}{\frac{1}{f}}$ to reduce it to $\frac{0}{0}, \frac{\infty}{\infty}$.

**Examples:**

$\boxed{7}$ $\displaystyle\lim_{x\to\infty} \sqrt{x}\, e^{-\frac{x}{2}} = \infty\cdot 0 \rightsquigarrow$

$$\lim_{x\to\infty} \sqrt{x}\, e^{-\frac{x}{2}} = \lim_{x\to\infty} \frac{\sqrt{x}}{e^{x/2}} = \lim_{x\to\infty} \frac{\frac{1}{2\sqrt{x}}}{\frac{1}{2}e^{x/2}} = \lim_{x\to\infty} \frac{1}{\sqrt{x}\, e^{x/2}} = 0.$$

$\boxed{8}$ $\displaystyle\lim_{x\to 0^+} \sin x \ln x = 0.(-\infty)$

$$\lim_{x\to 0^+} \frac{\ln}{(\sin x)^{-1}} \overset{H}{=} \frac{1/x}{-(\sin x)^{-2}\cdot\cos x} = \lim_{x\to 0} \frac{\sin^2 x}{-x\cos x}$$

$$\overset{H}{=} \lim_{x\to 0^+} \frac{2\sin x\cos x}{-\cos x + x\sin x} = 0$$

$\boxed{9}$ $\displaystyle\lim_{x\to 0^+} \ln x \tan\left(\frac{\pi x}{2}\right) = (-\infty)(0)$

$$\lim \frac{\ln x}{\tan(\frac{\pi}{2}x)^{-1}} \overset{H}{=} \frac{\frac{1}{x}}{-1(\tan(\frac{\pi}{2}x))^{-2}\,\frac{\pi}{2}\sec^2(\frac{\pi}{2}x)} = \frac{\tan^2(\frac{\pi}{2}x)}{x\,\frac{-\pi}{2}\sec^2(\frac{\pi}{2}x)}$$

$$= 0$$

## 3- Indeterminate Differences

$$\lim_{x\to\infty} f(x) - g(x) = \infty - \infty$$ "indeterminate form of type $\infty - \infty$"

$\boxed{10}$ $$\lim_{x\to\infty} (x - \ln x) = \lim_{x\to\infty} \ln e^x - \ln x = \lim_{x\to\infty} \ln \frac{e^x}{x}$$

$$= \ln \lim_{x\to\infty} \frac{e^x}{x} = \infty.$$

$\boxed{11}$ $$\lim_{x\to 1^+} \left( \frac{x}{x-1} - \frac{1}{\ln x} \right)$$

## 4- Indeterminate Powers

$$\lim_{x\to a} (f(x))^{g(x)}$$

Indeterminate type

1. $0^0$
2. $\infty^0$
3. $1^\infty$

Idea:

$$L = \lim_{x\to a} f(x)^{g(x)}$$

$$\ln L = \lim_{x\to a} g(x) \ln f(x).$$

$\boxed{12}$ $\lim\limits_{x\to 0^+} x^{\sqrt{x}} = 0^0$

$L = \lim\limits_{x\to 0^+} x^{\sqrt{x}} \longrightarrow \ln L = \lim\limits_{x\to 0^+} \sqrt{x}\,\ln x$

$= \lim\limits_{x\to 0^+} \dfrac{\ln x}{\frac{1}{\sqrt{x}}} = \lim\limits_{x\to 0^+} \dfrac{\ln x}{x^{-\frac{1}{2}}}$

$= \lim\limits_{x\to 0^+} \dfrac{\frac{1}{x}}{-\frac{1}{2}x^{-\frac{3}{2}}}$ $\cancel{= \lim\limits_{x\to\infty} \dfrac{-2}{x^{-\frac{5}{2}}} = -\infty.}$

$= \lim\limits_{x\to 0^+} \dfrac{-2\,x^{\frac{3}{2}}}{x} = \lim\limits_{x\to 0^+} -2x^{\frac{1}{2}} \Rightarrow \ln L = 0$

So $L = 1$

$\boxed{13}$ $\lim\limits_{x\to\infty} x^{\frac{1}{x}} = 0^{\infty}$

$L = \lim\limits_{x\to\infty} x^{\frac{1}{x}} \Rightarrow \ln L = \lim\limits_{x\to\infty} \frac{1}{x}\ln x = \lim\limits_{x\to\infty} \dfrac{\ln x}{x}$

$\overset{H}{=} \lim\limits_{x\to\infty} \frac{1}{x} \Rightarrow \ln L = 0 \Rightarrow L = 1$

$\boxed{14}$ $\lim\limits_{x\to\infty} (e^x + x)^{\frac{1}{x}} = 1^0$

$L = \lim\limits_{x\to\infty} (e^x + x)^{\frac{1}{x}} \Rightarrow \ln L = \lim\limits_{x\to\infty} \frac{1}{x}\ln(e^x + x) = \dfrac{\ln(e^x+x)}{x}$

$\ln L \overset{H}{=} \lim\limits_{x\to\infty} \dfrac{e^x+1}{e^x+x}$

$\overset{H}{=} \lim\limits_{x\to\infty} \dfrac{e^x}{e^x+1} = \dfrac{\infty}{\infty}$ $\Bigg\{$ $\overset{H}{=} \lim\limits_{x\to\infty} \dfrac{e^x}{e^x} = 1$

$L = e$