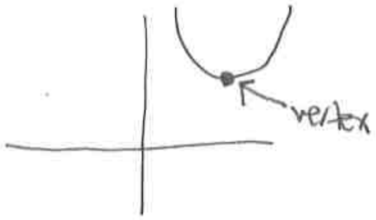
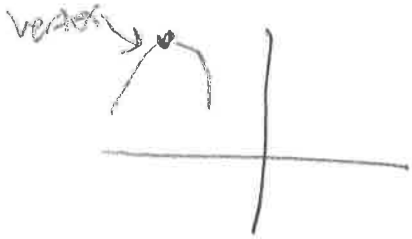
Appendum to § 5.6

1 To sketch the graph of the parabola $y = ax^2 + bx + c$

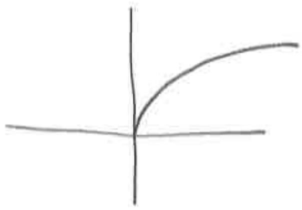
1 If $a > 0$, then the parabola is upward.
If $a < 0$, then the parabola is downward.

2 Find the vertex point $\left(\frac{-b}{2a}, f\left(-\frac{b}{2a}\right)\right)$.

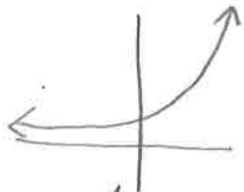




2 To sketch the graph of $y = \sqrt{x}$, it is of the general shape of



3 To sketch the graph of $y = a^x$, it is either



$a > 1$

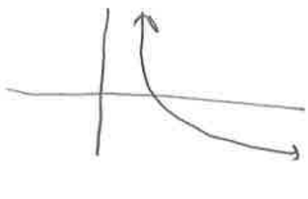
or

$0 < a < 1$

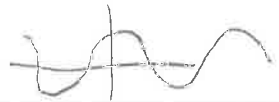
4 To sketch the graph of $y = \log_a x$, it is either

$a > 1$

or

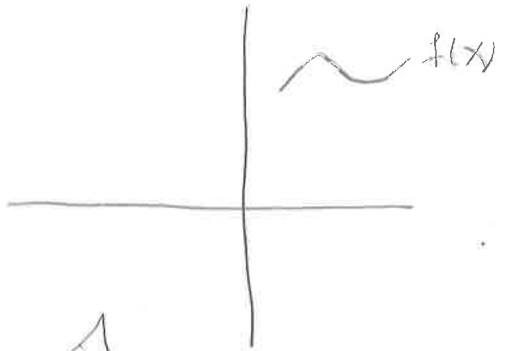


5 $y = \sin x$, $y = \cos x$



Curve Sketching

- If you know the graph of the basic functions, then



$y = f(x+c)$
"Shift the graph c units to the left"

$y = f(x) + c$
"Shift the graph c units upward

$y = f(x) - c$
"Shift the graph c unit downward"

$y = f(x-c)$
"Shift the graph c units to the right"

- $y = cf(x)$, stretch the graph <u>vertically</u> by a factor of c.
- $y = f(cx)$, strink the graph horizontally by a factor of c.
- $y = -f(x)$, reflect the graph about the x-axis.
- $y = f(-x)$, reflect the graph about the y-axis.

<u>Exercise</u> : Sketch

- $y = \sqrt{x-2} + 3$
- $y = \cos 2x$
- $y = -\sin x$
- $y = e^{-x}$
- $y = 1 - \sin x$
- $y = \sin \frac{1}{2} x$.

- To sketch the graph of $x = g(y)$

write y in terms of x ~~to~~ get

$$y = g^{-1}(x)$$

and sketch its graph

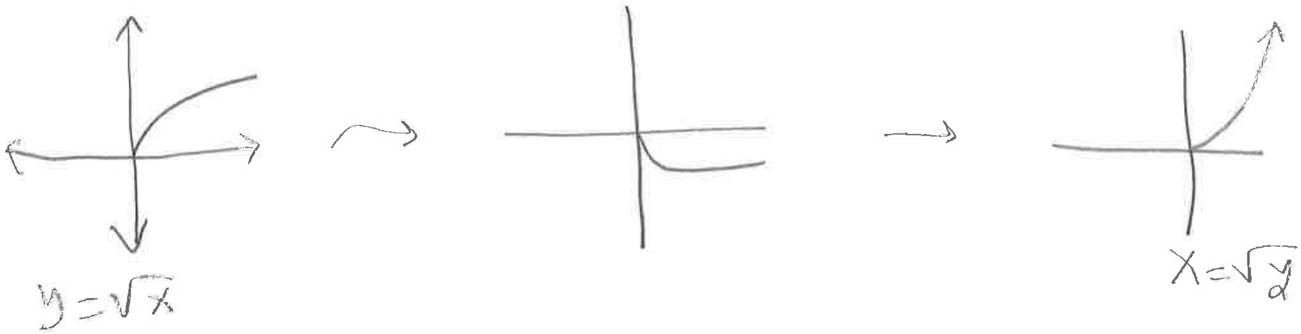
Sketch the graph of g as $y = g(x)$ and then

1- reflect on the x-axis.

2- rotate 45° counterclockwise

same as reflecting on the line $y = x$.

Example : sketch $x = \sqrt{y}$.

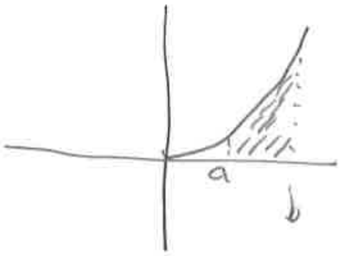


Exercise : sketch the graph of $x = (y-2)^2 + 1$.

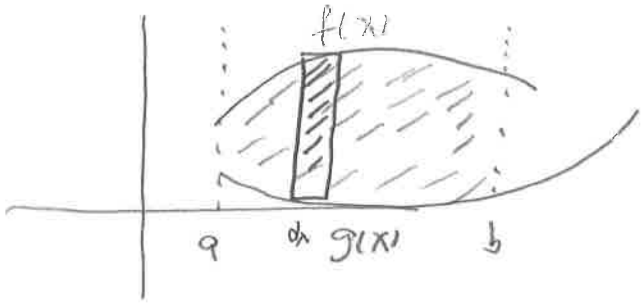
§ 5.6 - Areas Between Curves

Recall:

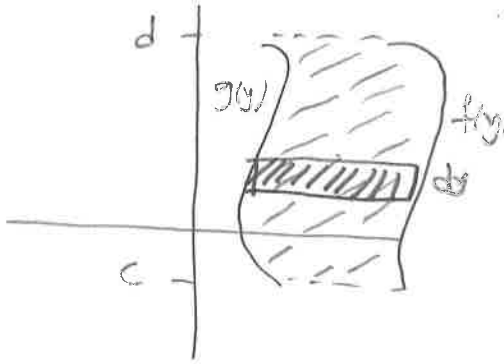
$$\int_a^b f(x)\,dx = \text{Area under the graph of } f \text{ on the interval } [a,b].$$



Goal: To find the area between curves, e.g.;



or



We use the idea of placing a rectangle in the region.

Area of the rectangle = length × width

$= (\underset{\text{Top}}{f(x)} - \underset{\text{bottom}}{g(x)})\,dx$ or $= (\underset{\text{right}}{f(y)} - \underset{\text{left}}{g(y)})\,dy$

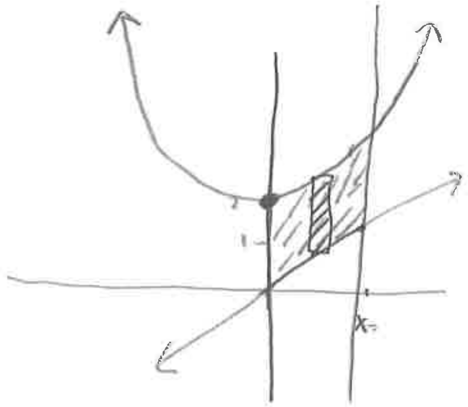
Total Area $= \int_a^b (f(x)-g(x))\,dx$ ← must be positive!

or $\int_c^d (f(y)-g(y))\,dy$.

Examples10: Find the area of the region bounded by

$\boxed{1}$ $y = x^2 + 2$, $y = x$, $x = 0$, $x = 1$

we first sketch the region first and then we place the rectangle.

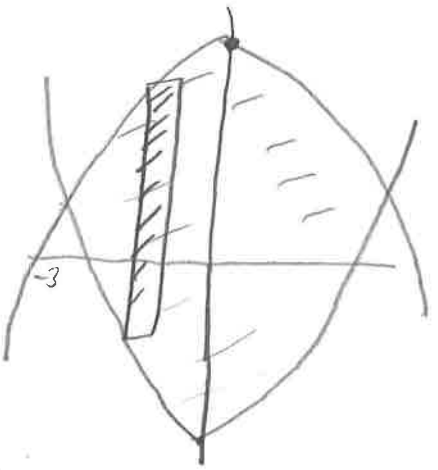


area of the rectangle $= \Big[\underbrace{(x^2+2)}_{\text{Top}} - \underbrace{(x)}_{\text{bottom}}\Big]\, dx$

$$\text{Area} = \int_0^1 (x^2+2-x)\,dx = \left[\frac{x^3}{3} + 2x - \frac{x^2}{2}\right]_0^1 = \frac{11}{6}.$$

$\boxed{2}$ $y = 12 - x^2$, $y = x^2 - 6$

vertex $= (0, 12)$ vertext $= (0, -6)$



we need to find the points of intersections, so we solve both equations to get $12 - x^2 = x^2 - 6 \rightarrow 2x^2 = 18 \rightarrow x = \pm 3$

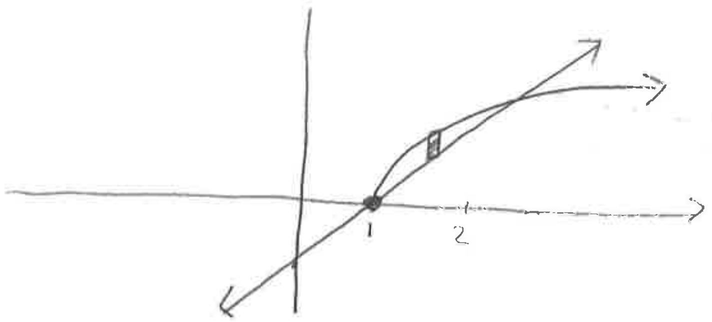
So we have

$$\text{Area of the rectangle} = \Big[(\underbrace{12 - x^2}_{\text{Top}}) - (\underbrace{x^2 - 6}_{\text{Bottom}}) \Big] dx$$

$$\text{Area} = \int_{-3}^{3} (12 - x^2 - x^2 + 6) \, dx = \int_{-3}^{3} (18 - 2x^2) \, dx$$

$$= \Big[18x - \frac{2}{3} x^3 \Big]_{-3}^{3} = \underline{\underline{72}}$$

$\boxed{3}$ $y = \sqrt{x-1}$, $x - y = 1 \rightarrow y = x - 1$



we find the point of intersection first, so we have to solve $\sqrt{x-1} = x - 1 \longrightarrow x - 1 = (x-1)^2 \longrightarrow x - 1 = x^2 - 2x + 1$

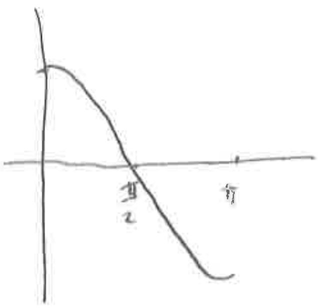
$0 = x^2 - 3x - 2 \longrightarrow x = 1, 2$.

$$\text{Area of the rectangle} = \Big[(\sqrt{x-1}) - (x-1) \Big] dx$$

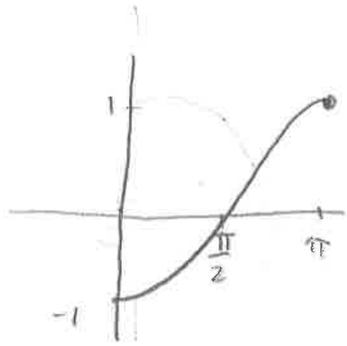
$$\text{Area} = \int_{\phi}^{2} (\sqrt{x-1}) - (x-1)) \, dx = \int_{1}^{2} ((x-1)^{\frac{1}{2}} - x + 1) \, dx$$

$$= \Big[\frac{2}{3} (x-1)^{\frac{3}{2}} - \frac{x^2}{2} + x \Big]_{1}^{2} = \underline{\underline{\frac{1}{6}}}$$

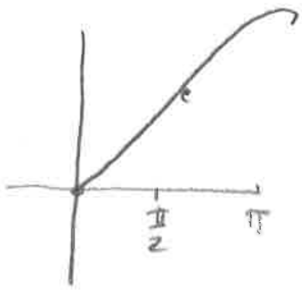
4 $y = \cos x, \quad y = 1-\cos x, \quad 0 \le x \le \pi$



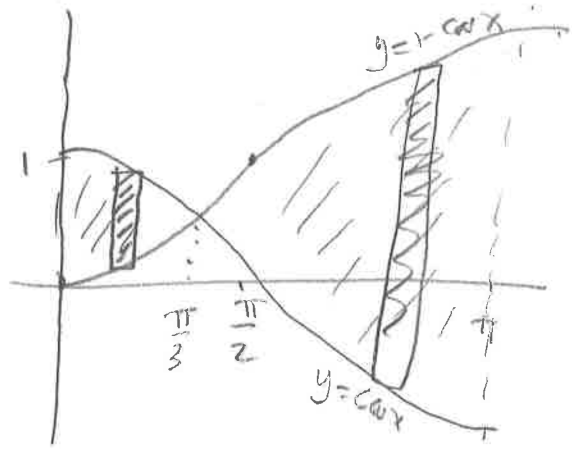
$y=\cos x$



$y = -\cos x$



$y = 1-\cos x$



we find the points of intersection, so we solve

$$\cos x = 1-\cos x \rightarrow \cos x = \frac{1}{2} \rightarrow x = \frac{\pi}{3}$$

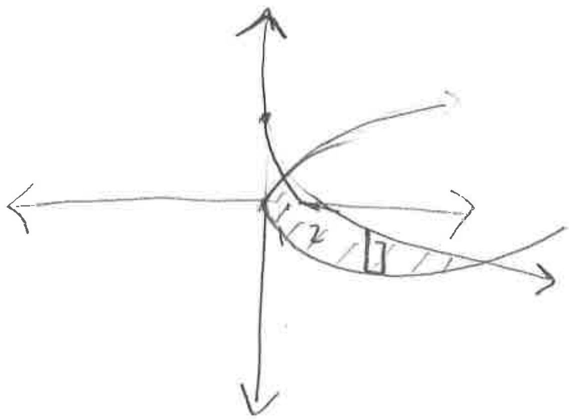
$$\text{Area of rectangle} = (\cos x-(1-\cos x))\,dx \quad + \quad (1-\cos x - \cos x)$$

$$\text{Total area} = \int_0^{\frac{\pi}{3}} (\cos x-(1-\cos x))\,dx + \int_{\frac{\pi}{3}}^{\pi} (1-2\cos x)\,dx$$

$$= \underbrace{\cdots}_{\text{(exercise)}}$$

6 $X = y^4$, $y = 2 + \sqrt{-x}$, $y = 0$

$y = 2 + \sqrt{-x}$, $y = 0$



we find the points of intersections, so $\sqrt[4]{x} = \sqrt{2-x}$

$X = (2-x)^2 \rightarrow X = 4 - 4X + X^2 \rightarrow X^2 - 5X + 4 = 0$

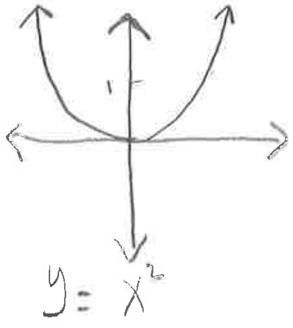
$X = +4$, $X = +1$

$$\text{Area} = \int_1^4 \left[\sqrt{2-x} - \underbrace{\sqrt[4]{x}}_{y}\right] dx$$

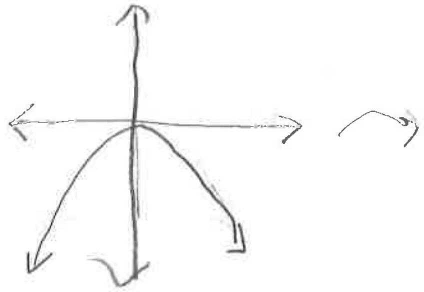
7 Exercise $y = \sin x$, $y = e^x$, $X = 0$, $X = \frac{\pi}{2}$.

$\boxed{5}$ $4x + y^2 = 12$, $x = y$

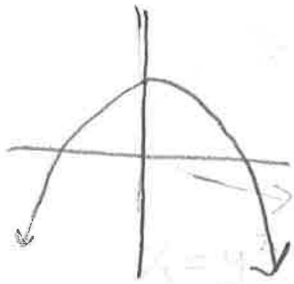
$$x = \frac{12 - 1y^2}{4} = 3 - \frac{1}{4}y^2$$



$y = x^2$

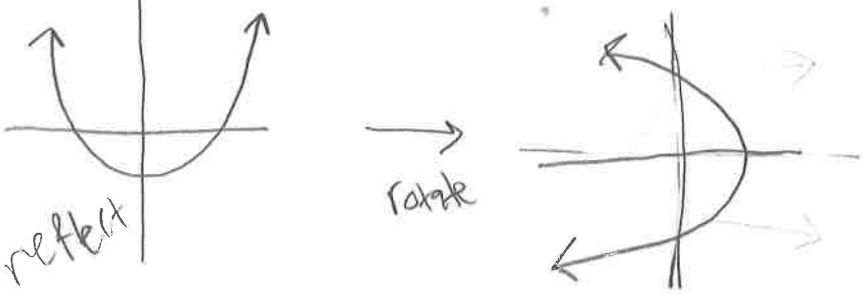


$x = -\frac{1}{4}y^2$



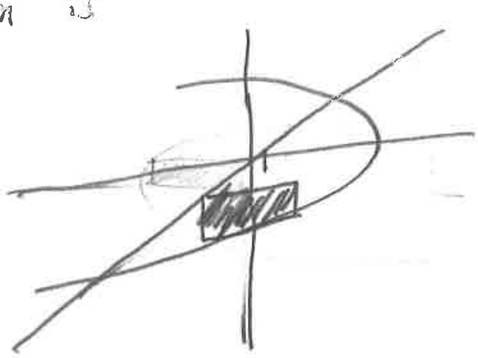
$x = 3 - \frac{1}{4}y^2$

Now



$x = 3 - \frac{1}{4}y^2$

So the region is



we find the points of intersection, which $y = 3 - \frac{1}{4}y^2$

So $y = 2$, $y = -6$. Now that

$$\text{Area} = \int_{-6}^{2} \left[\left(3 - \frac{1}{4}y^2\right) - y \right] dy$$