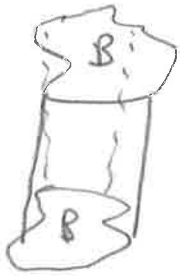
§ 6.1 - Volume

1- Definition of the volume

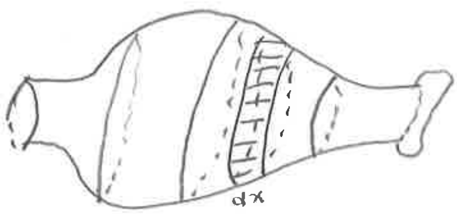
Definition:

The volume of a cylinder with base B and height h is

$$V := hA \,, \quad A \text{ is the area of the base}$$



In general solids,



we place a small cylinder with height dx and area $A(x)$

So

$$\text{Volume of cylinder} = A(x)\,dx$$

$$\text{Volume of the solid} = \int_a^b A(x)\,dx$$

2- Surface of Revolution

Definition:

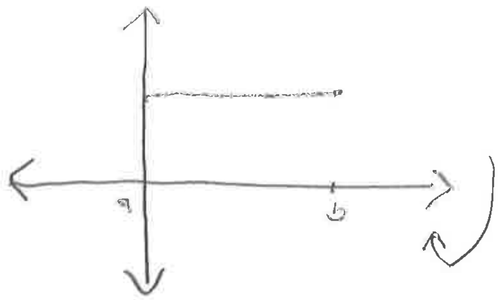
The surface of revolution is a 3-dimensional solid that is generating by rotating 2-dimensional curve around a straight line in a circular motion. The straight line is called the axes of revolution.

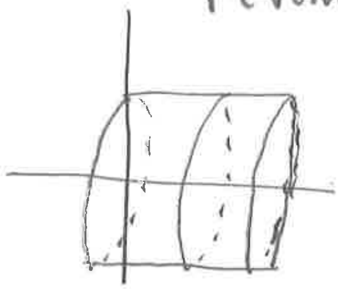
Example 1:

Curve

Surface of revolution

A.

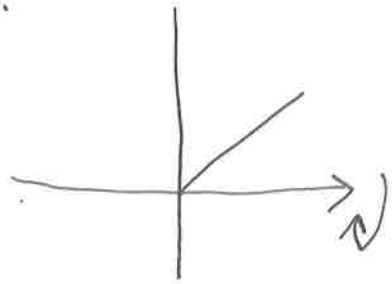


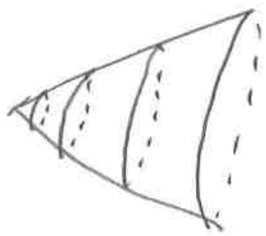


axis of revolution is $y=0$

Circular Cylinder

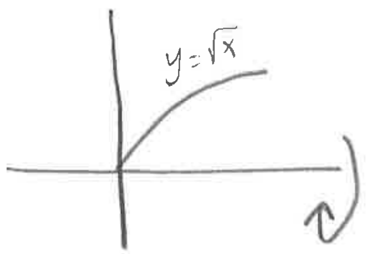
B.

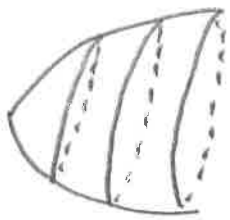




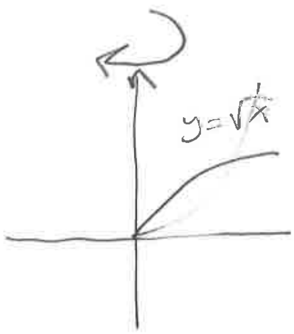
Circular cone.

C.

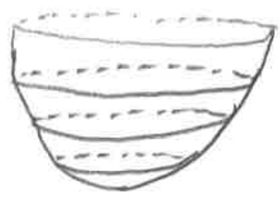




D.

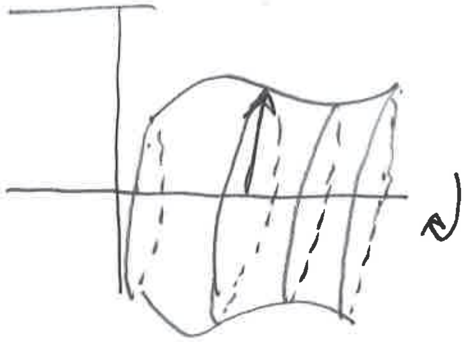


E.



3- Finding volume of Surface of revolution

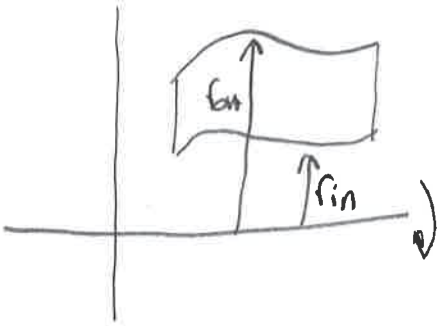
Case 1: Disk



we place a small ~~Disk~~ (cylinder) of height dx and we find the area

$$\text{Volume} = (\underbrace{\pi r^2}_{\text{area of circle}})\, dx$$

Case 2: Washer

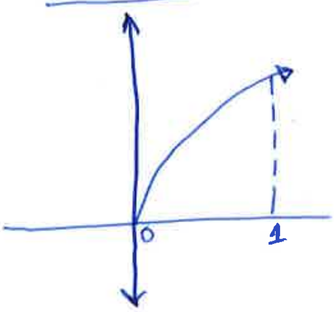


$$\text{Volume of a washer} = \left(\pi (r_{out})^2 - \pi (r_{in})^2\right) dx$$

and then we use integration to find the total volume.

Example 2: Find the volume of the solid obtained by rotating the curve $y = \sqrt{x}$ around the x-axis from $x=0$ to $x=1$

Solution:



$r_{out} = \text{Top} - \text{bottom}$

$= \sqrt{x} - 0$

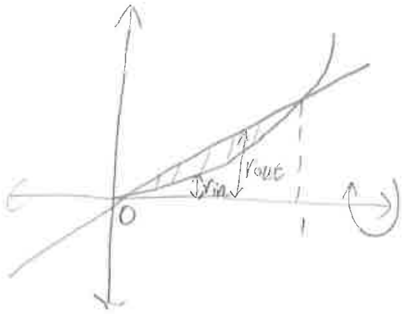
$= \sqrt{x}$

$$\text{Volume} = \int_0^1 \pi (\sqrt{x})^2 dx$$

$$= \int_0^1 \pi x\, dx$$

$$= \pi \left[\frac{x^2}{2}\right]_0^1 = \frac{\pi}{2}$$

Example 2: Find the volume of the surface of revolution of the region enclosed by $y=x$, $y=x^2$ around the x-axis.



$r_{out}=x$ / $r_{in}=x^2$

$$Volume=\int_a^b \pi(r_{out})^2-\pi(r_{in})^2\,dx$$

$$=\pi\int_0^1(x^2-x^4)dx=\pi\left[\frac{x^3}{3}-\frac{x^5}{5}\right]\Bigg|_0^1$$

$$=\frac{2\pi}{15}$$

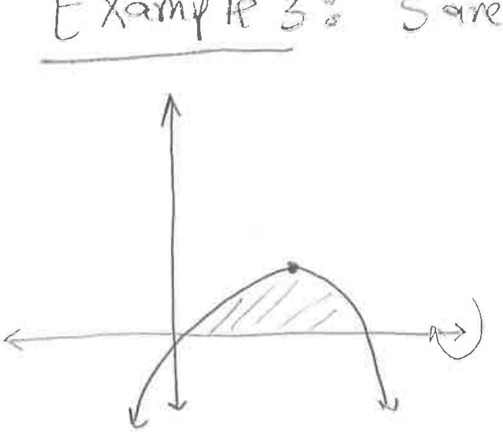
$x=x^2$

$x^2-x=0$

$x_1=0$

$x_2=1$

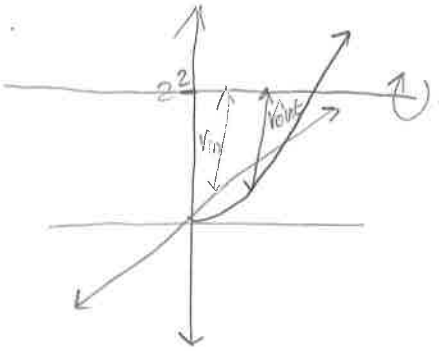
Example 3: Same as above, $y=x-x^2$, $y=0$, around x-axis



$r_{out}=x-x^2$

$$Volume=\int_0^1\pi(x-x^2)^2\,dx=\frac{\pi}{30}$$

Example 4: $y = x$, $y = x^2$, around $y = 2$

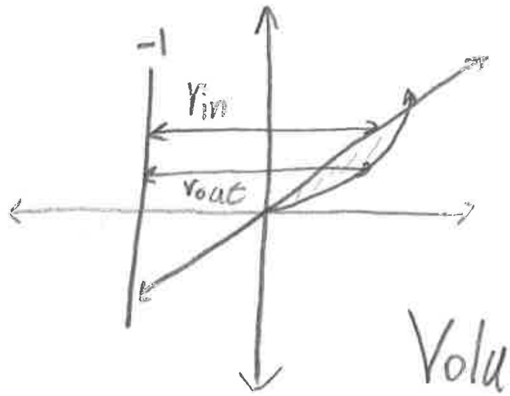


$r_{out} = 2 - x^2$

$r_{in} = 2 - x$

$V = \pi \int_0^1 (2-x^2)^2 - (2 - x)^2 \, dx =$

Example 5: $y = x$, $y = x^2$ around $x = -1$.



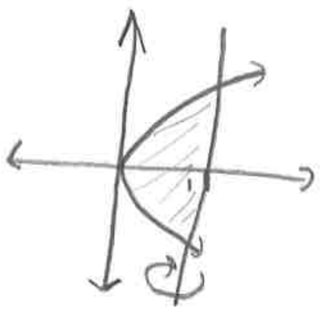
$r_{out} = \underbrace{\sqrt{y}}_{\text{in terms of } y} - (-1) = \sqrt{y} + 1$

$r_{in} = y - (-1) = y + 1$

$$\text{Volume} = \pi \int_0^1 (\sqrt{y} + 1)^2 - (y+1)^2 \, dy$$

$=$

Example 6: $x = y^2$, $x = 1$; about $x = 1$.

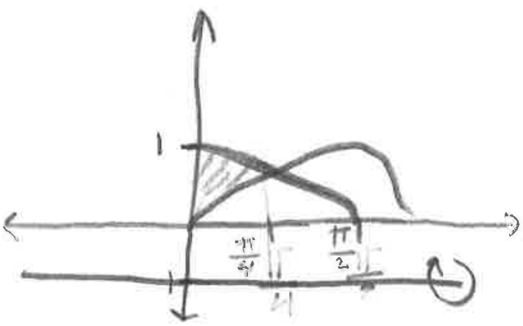


$r_{out} = 1 - y^2$

$r_{in} = 0$

$V = \int_{-1}^{1} \pi \left[(1-y^2)^2 \right] dy$

Example 7: $y = \sin x$, $y = \cos x$, $0 \le x \le \frac{\pi}{4}$; about $y = -1$.



$r_{out} = \cos x - (-1) = \cos x + 1$

$r_{in} = \sin x - (-1) = \sin x + 1$

$V = \pi \int_{0}^{\frac{\pi}{4}} \left[(\cos + 1)^2 - (\sin x + 1)^2 \right] dx$

Example 8: The region enclosed by the triangle with vertices (1, 0), (2, 1), and (1, 1) about $y=0$